OPTIMAL AND MAXIMAL SINGULAR CURVES

YVES AUBRY AND ANNAMARIA IEZZI

Abstract. Using an Euclidean approach, we prove a new upper bound for the number of closed points of degree 2 on a smooth absolutely irreducible projective algebraic curve defined over the finite field $\mathbb{F}_q$. This bound enables us to provide explicit conditions on q, g and π for the non-existence of absolutely irreducible projective algebraic curves defined over $\mathbb{F}_q$ of geometric genus g, arithmetic genus π and with $N_q(g)+\pi-g$ rational points. Moreover, for q a square, we study the set of pairs (g,π) for which there exists a maximal absolutely irreducible projective algebraic curve defined over $\mathbb{F}_q$ of geometric genus g and arithmetic genus π, i.e. with $q+1+2g\sqrt{q}+\pi-g$ rational points.

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MSC[2010] 14H20, 11G20, 14G15.

1. Introduction

Throughout the paper[1], the word *curve* will stand for a (non-necessarily smooth) absolutely irreducible projective algebraic curve and $\mathbb{F}_q$ will denote the finite field with q elements.

Let X be a curve defined over $\mathbb{F}_q$ of geometric genus g and arithmetic genus π. The first author and Perret showed in [4] that the number $\sharp X(\mathbb{F}_q)$ of rational points over $\mathbb{F}_q$ on X satisfies:

$$\sharp X(\mathbb{F}_q) \leq q+1+g[2\sqrt{q}]+\pi-g. \tag{1}$$

Furthermore, if we denote by $N_q(g,\pi)$ the maximum number of rational points on a curve defined over $\mathbb{F}_q$ of geometric genus g and arithmetic genus π, it is proved in [2] that:

$$N_q(g) \leq N_q(g,\pi) \leq N_q(g)+\pi-g,$$

where $N_q(g)$ classically denotes the maximum number of rational points over $\mathbb{F}_q$ on a smooth curve defined over $\mathbb{F}_q$ of genus g.

The curve X is said to be *maximal* if it attains the bound (1). This definition for non-necessarily smooth curves has been introduced in [2] and extends the classical definition of maximal curve when X is smooth.

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More generally (see [2]), X is said to be *δ-optimal* if

$$\sharp X(\mathbb{F}_q) = N_q(g) + \pi - g.$$

Obviously, the set of maximal curves is contained in that of δ-optimal ones.

In [2], we were interested in the existence of δ-optimal and maximal curves of prescribed geometric and arithmetic genera. Precisely, we proved (see Theorem 5.3 in [2]):

$$N_q(g,\pi) = N_q(g) + \pi - g \iff g \leq \pi \leq g + B_2(\mathcal{X}_q(g)), \qquad (2)$$

where $\mathcal{X}_q(g)$ denotes the set of optimal smooth curves defined over $\mathbb{F}_q$ of genus g (i.e. with $N_q(g)$ rational points) and $B_2(\mathcal{X}_q(g))$ the maximum number of closed points of degree 2 on a curve of $\mathcal{X}_q(g)$.

The quantity $B_2(\mathcal{X}_q(g))$ is easy to compute for g equal to 0 and 1 and also for those g for which $N_q(g) = q + 1 + g[2\sqrt{q}]$ (see Corollary 5.4, Corollary 5.5 and Proposition 5.8 in [2]), but is not explicit in the general case.

The first aim of this paper is to provide upper and lower bounds for $B_2(\mathcal{X}_q(g))$. For this purpose, we will follow the Euclidean approach developed by Hallouin and Perret in [10] and recalled in Section 2. These new bounds will allow us to provide explicit conditions on q, g and π for the non-existence of δ-optimal curves and to determine some exact values of $N_q(g,\pi)$ for specific triples (q, g, π).

Secondly, in Section 4, we will assume q to be square and, as in the smooth case, we will study the genera spectrum of maximal curves defined over $\mathbb{F}_q$, i.e. the set of pairs (g,π), with $g, \pi \in \mathbb{N}$ and $g \leq \pi$, for which there exists a maximal curve defined over $\mathbb{F}_q$ of geometric genus g and arithmetic genus π.

2. Hallouin-Perret's approach

Let X be a smooth curve defined over $\mathbb{F}_q$ of genus $g > 0$.

For every positive integer n, we associate to X a n-tuple $(x_1, \ldots, x_n)$ defined as follows:

$$x_i := \frac{(q^i + 1) - \sharp X(\mathbb{F}_{q^i})}{2g\sqrt{q^i}}, \quad i = 1, \ldots, n. \qquad (3)$$

The Riemann Hypothesis, proved by Weil in positive characteristic, gives that

$$\sharp X(\mathbb{F}_{q^i}) = q^i + 1 - \sum_{j=1}^{2g} \omega_j^i, \qquad (4)$$

where $\omega_1, \ldots, \omega_{2g}$ are complex numbers of absolute value $\sqrt{q}$. Hence one easily gets $|x_i| \leq 1$ for all $i = 1, \ldots, n$, which means that the n-tuple $(x_1, \ldots, x_n)$ belongs to the hypercube

$$\mathcal{C}_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | -1 \leq x_i \leq 1, \ \forall\, i = 1, \ldots, n\}. \qquad (5)$$

The Hodge Index Theorem implies that the intersection pairing on the Neron-Severi space over $\mathbb{R}$ of the smooth algebraic surface $X \times X$ is anti-Euclidean on the orthogonal complement of the trivial plane generated by the horizontal and vertical classes. Hallouin and Perret used this fact in [10] to obtain that the following matrix

$$G_n = \begin{pmatrix} 1 & x_1 & \cdots & x_{n-1} & x_n \\ x_1 & 1 & x_1 & \ddots & x_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ x_{n-1} & \ddots & \ddots & 1 & x_1 \\ x_n & x_{n-1} & \cdots & x_1 & 1 \end{pmatrix}$$

is a Gram matrix and thus is positive semidefinite (the x_i's are interpreted as inner products of normalized Neron-Severi classes of the iterated Frobenius morphisms).

Now, a matrix is positive semidefinite if and only if all the principal minors are non-negative. This fact implies that the n-tuple $(x_1, \ldots, x_n)$ has to belong to the set

$$\mathcal{W}_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n |\, G_{n,I} \geq 0, \forall\, I \subset \{1, \ldots, n+1\}\}, \qquad (6)$$

where $G_{n,I}$ represents the principal minor of G_n obtained by deleting the rows and columns whose indexes are not in I.

To these relations, which come from the geometrical point of view, one can add the arithmetical contraints resulting from the obvious following inequalities pointed by Ihara in [11]: $\sharp X(\mathbb{F}_{q^i}) \geq \sharp X(\mathbb{F}_q)$, for all $i \geq 2$. It follows that, for all $i \geq 2$,

$$x_i \leq \frac{x_1}{q^{\frac{i-1}{2}}} + \frac{q^{i-1} - 1}{2gq^{\frac{i-2}{2}}}.$$

Setting

$$h_i^{q,g}(x_1, x_i) = x_i - \frac{x_1}{\sqrt{q}^{i-1}} - \frac{\sqrt{q}}{2g}\left(\sqrt{q}^{i-1} - \frac{1}{\sqrt{q}^{i-1}}\right)$$

one gets that the n-tuple $(x_1, \ldots, x_n)$ has to belong to the set

$$\mathcal{H}_n^{q,g} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | h_i^{q,g}(x_1, x_i) \leq 0, \text{ for all } 2 \leq i \leq n\}. \qquad (7)$$

For convenience, we set $\mathcal{H}_1^{q,g} = \mathbb{R}$.

Remark 2.1. We have $h_i^{q,g}(x_1, x_i) = 0$ if and only if $\sharp X(\mathbb{F}_q) = \sharp X(\mathbb{F}_{q^i})$.

Finally we obtain (Proposition 16 in [10]) that if X is a smooth curve defined over $\mathbb{F}_q$ of genus $g > 0$, then its associated n-tuple $(x_1, \ldots, x_n)$ belongs to $\mathcal{C}_n \cap \mathcal{W}_n \cap \mathcal{H}_n^{q,g}$, where $\mathcal{C}_n, \mathcal{W}_n, \mathcal{H}_n^{q,g}$ are respectively defined by (5), (6) and (7).

Fixing $n = 1, 2, 3, \ldots$, we find compact subsets of $\mathbb{R}^n$ to which the n-tuple $(x_1, \ldots, x_n)$ belongs. Hence we can obtain lower or upper bounds for $\sharp X(\mathbb{F}_{q^i})$ by noting that any lower bound for x_i corresponds to an upper bound for $\sharp X(\mathbb{F}_{q^i})$ and, vice versa, any upper bound for x_i corresponds to a lower bound for $\sharp X(\mathbb{F}_{q^i})$.

Hallouin and Perret showed in [10] that, increasing the dimension n, the set $\mathcal{C}_n \cap \mathcal{W}_n \cap \mathcal{H}_n^{q,g}$ provides an increasingly sharp lower bound for x_1 (and hence an increasingly sharp upper bound for $\sharp X(\mathbb{F}_q)$) if g is large enough compared to q.

Indeed, they first recovered, for $n = 1$, the classical Weil bound, that can be seen as a *first-order Weil bound*:

$$\sharp X(\mathbb{F}_q) \leq q + 1 + 2g\sqrt{q}.$$

For $n = 2$, they recovered the Ihara bound proved in [11] (to which they refered as the *second-order Weil bound*): if $g \geq g_2 := \frac{\sqrt{q}(\sqrt{q}-1)}{2}$ then

$$\sharp X(\mathbb{F}_q) \leq q + 1 + \frac{\sqrt{(8q+1)g^2 + 4q(q-1)g} - g}{2}.$$

And for $n = 3$, they found a *third-order Weil bound* for $g \geq g_3 := \frac{\sqrt{q}(q-1)}{\sqrt{2}}$ as stated in Theorem 18 of [10]. But, thanks to Ivan Semeniuk who worked on this question in his Master thesis, it appears that for some values of q and $g \geq g_3$, this third-order Weil bound is not better than the second order one, and this implies that the bound given by Hallouin and Perret in Theorem 18 of [10] is not correct. We corrected the bound and finally we find that, for $g \geq g_3$, we have:

$$\sharp X(\mathbb{F}_q) \leq q + 1 + \left(\frac{\sqrt{a(q) + \frac{b(q)}{g} + \frac{c(q)}{g^2}} - 1 - \frac{1}{q} - \frac{1}{g}d(q)}{1 + \frac{2}{\sqrt{q}}} \right) g\sqrt{q},$$

where

$$\begin{cases} a(q) = 5 + \frac{8}{\sqrt{q}} + \frac{2}{q} + \frac{1}{q^2} \\ b(q) = \frac{(q^2-1)(3\sqrt{q}-1)(\sqrt{q}+1)}{q\sqrt{q}} \\ c(q) = \frac{(q-1)^2(-4q^{\frac{3}{2}} - 4q^{\frac{1}{2}} + q^2 - 2q + 1)}{4q} \\ d(q) = \frac{(q-1)(q-2\sqrt{q}-1)}{2\sqrt{q}}. \end{cases}$$

In a similar way, we would like to find increasingly sharp lower bounds for x_2 (possibly depending on x_1), in order to provide new upper bounds for $\sharp X(\mathbb{F}_{q^2})$. From each of these bounds we will deduce a new upper bound for the number of closed points of degree 2 on X and hence we will be able to make our equivalence (2) more explicit.

3. Number of closed points of degree 2

Let X be a smooth curve defined over $\mathbb{F}_q$ of genus g. We recall that, if $B_2(X)$ denotes the number of closed points of degree 2 on X, one has

$$B_2(X) = \frac{\sharp X(\mathbb{F}_{q^2}) - \sharp X(\mathbb{F}_q)}{2}.$$

3.1. **Upper bounds.** We are going to establish upper bounds for the number $B_2(X)$ and then obtain upper bounds for the quantity $B_2(\mathcal{X}_q(g))$ defined as the maximum number of closed points of degree 2 on an optimal smooth curve of genus g defined over $\mathbb{F}_q$.

3.1.1. *First order.* From the Weil bounds related to (4), we get $\sharp X(\mathbb{F}_{q^2}) \leq q^2 + 1 + 2gq$ and $\sharp X(\mathbb{F}_q) \geq q + 1 - 2g\sqrt{q}$. Hence an obvious upper bound for $B_2(X)$ is:

$$B_2(X) \leq \frac{q^2 - q}{2} + g(q + \sqrt{q}) =: M'(q, g). \tag{8}$$

We can consider $M'(q, g)$ as an upper bound for $B_2(\mathcal{X}_q(g))$ at the first order since this bound is a direct consequence of the Weil bounds.

Using the quantity $M'(q, g)$, we have recorded in the following table some first-order upper bounds for $B_2(\mathcal{X}_q(g))$ for specific pairs (q, g):

q \ g	2	3	4	5	6
2	7	11	14	18	21
3	12	17	21	26	31
2^2	18	24	30	36	42

Table 1. First-order upper bounds for $B_2(\mathcal{X}_q(g))$ given by $M'(q, g)$.

Unfortunately, the bound (8) is rather bad, so let us improve it.

We assume g to be positive and we consider $B_2(X)$ as a function of x_1 and x_2, defined in (3), in the domain $\mathcal{C}_n \cap \mathcal{W}_n \cap \mathcal{H}_n^{q,g}$ to which x_1 and x_2 belong:

$$B_2(X) = g\sqrt{q}(x_1 - \sqrt{q}x_2) + \frac{q^2 - q}{2} \tag{9}$$

since $\sharp X(\mathbb{F}_q) = q + 1 - 2g\sqrt{q}x_1$ and $\sharp X(\mathbb{F}_{q^2}) = q^2 + 1 - 2gqx_2$.

We note that any lower bound for x_2 implies an upper bound for $B_2(X)$, possibly depending on x_1.

We are going to investigate the set $\mathcal{C}_n \cap \mathcal{W}_n \cap \mathcal{H}_n^{q,g}$ introduced in the previous section for $n = 2$ (second order) and $n = 3$ (third order).

3.1.2. *Second order.* For $n = 2$ the set $\mathcal{C}_2 \cap \mathcal{W}_2 \cap \mathcal{H}_2^{q,g}$ is given by the pairs $(x_1, x_2) \in \mathbb{R}^2$ which satisfy the following system of inequalities:

$$\begin{cases} 2x_1^2 - 1 \leq x_2 \leq 1 \\ x_2 \leq \frac{x_1}{\sqrt{q}} + \frac{q-1}{2g}. \end{cases} \tag{10}$$

Geometrically, it corresponds to the region of the plane $\langle x_1, x_2 \rangle$ delimited by the parabola $P : x_2 = 2x_1^2 - 1$ and the lines $L_2^{q,g} : x_2 = \frac{x_1}{\sqrt{q}} + \frac{q-1}{2g}$ and $x_2 = 1$. More precisely, depending on whether $g < g_2$, $g = g_2$ or $g > g_2$, where $g_2 = \frac{\sqrt{q}(\sqrt{q}-1)}{2}$, the region can assume one of the following three configurations:

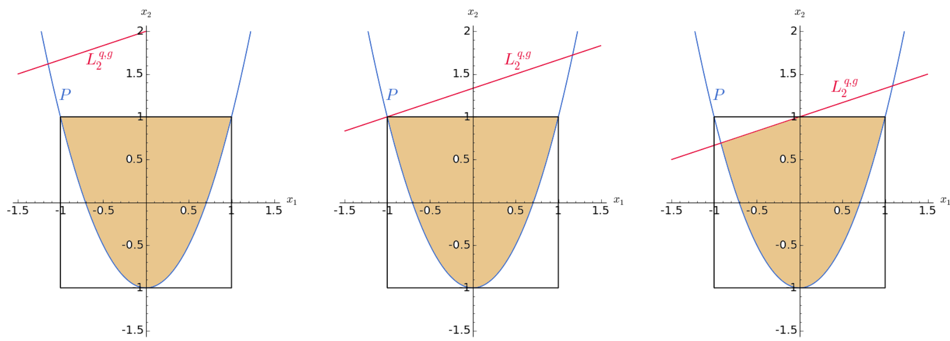


TABLE 2. The region $\mathcal{C}_2 \cap \mathcal{W}_2 \cap \mathcal{H}_2^{q,g}$, respectively for $g < g_2$, $g = g_2$ and $g > g_2$.

The first inequality in the system (10)

$$x_2 \geq 2x_1^2 - 1, \tag{11}$$

yields the upper bound:

$$B_2(X) \leq g\sqrt{q}(x_1 - \sqrt{q}(2x_1^2 - 1)) + \frac{q^2 - q}{2}. \tag{12}$$

Using equation (3) for x_1, we get the following bound for $B_2(X)$ as a function of q, g and $\sharp X(\mathbb{F}_q)$, which is a reformulation of Proposition 14 of [10]:

Proposition 3.1. *Let X be a smooth curve of genus $g > 0$ over $\mathbb{F}_q$. We have:*

$$B_2(X) \leq \frac{q^2 + 1 + 2gq - \frac{1}{g}\left(\sharp X(\mathbb{F}_q) - (q+1)\right)^2 - \sharp X(\mathbb{F}_q)}{2}.$$

Now let us assume that X is an optimal smooth curve of genus $g > 0$, that is X has $N_q(g)$ rational points. By Proposition 3.1, if we set

$$M''(q, g) := \frac{q^2 + 1 + 2gq - \frac{1}{g}\left(N_q(g) - (q+1)\right)^2 - N_q(g)}{2},$$

then we have:

$$B_2(\mathcal{X}_q(g)) \leq M''(q,g).$$

The quantity $M''(q,g)$ can hence be seen as a second-order upper bound for $B_2(\mathcal{X}_q(g))$.

We obtain the following proposition, as an easy consequence of (2):

Proposition 3.2. *Let $g > 0$. If $\pi > g + M''(q,g)$, then no δ-optimal curve defined over $\mathbb{F}_q$ of geometric genus g and arithmetic genus π exist.*

In the following table, we have used the quantity $M''(q,g)$ to get upper bounds for $B_2(\mathcal{X}_q(g))$ for specific pairs (q,g) (we used the data about $N_q(g)$ available in [15]).

q \ g	2	3	4	5	6
2	1	2	3	4	5
3	3	3	3	5	7
2^2	5	0	4	5	3

TABLE 3. Second-order upper bounds for $B_2(\mathcal{X}_q(g))$ given by $M''(q,g)$.

3.1.3. *Third order.* If we now increase the dimension to $n = 3$, new constraints for x_1, x_2, x_3 arise in addition to those of the system (10). Indeed, the set $\mathcal{C}_3 \cap \mathcal{W}_3 \cap \mathcal{H}_3^{q,g}$ is given by the triples $(x_1, x_2, x_3) \in \mathbb{R}^3$ which satisfy the following system of inequalities:

$$\begin{cases} 2x_1^2 - 1 \leq x_2 \leq 1 \\ -1 + \frac{(x_1+x_2)^2}{1+x_1} \leq x_3 \leq 1 - \frac{(x_1-x_2)^2}{1-x_1} \\ 1 + 2x_1x_2x_3 - x_3^2 - x_1^2 - x_2^2 \geq 0 \\ x_2 \leq \frac{x_1}{\sqrt{q}} + \frac{q-1}{2g} \\ x_3 \leq \frac{x_1}{q} + \frac{q^2-1}{2g\sqrt{q}}. \end{cases}$$

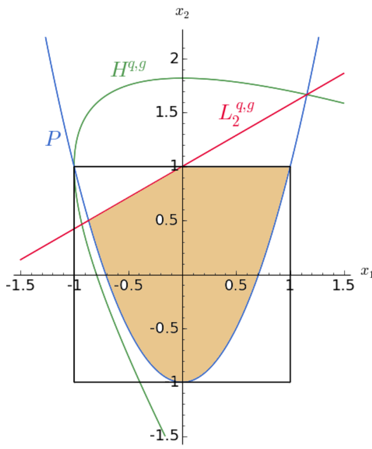
Let us consider the projection of $\mathcal{C}_3 \cap \mathcal{W}_3 \cap \mathcal{H}_3^{q,g}$ on the plane $\langle x_1, x_2 \rangle$, that is the set $\{(x_1,x_2) \in \mathbb{R}^2 : (x_1,x_2,x_3) \in \mathcal{C}_3 \cap \mathcal{W}_3 \cap \mathcal{H}_3^{q,g}\}$. It is easy to show that this set is given by the pairs $(x_1,x_2) \in \mathbb{R}^2$ which satisfy:

$$\begin{cases} 2x_1^2 - 1 \leq x_2 \leq 1 \\ -1 + \frac{(x_1+x_2)^2}{1+x_1} \leq \frac{x_1}{q} + \frac{q^2-1}{2g\sqrt{q}} \\ x_2 \leq \frac{x_1}{\sqrt{q}} + \frac{q-1}{2g}. \end{cases} \tag{13}$$

The equation which corresponds to the second inequality in the system (13):

$$x_2^2 + 2x_1x_2 - \left(\frac{1}{q} - 1\right)x_1^2 - \left(\frac{1}{q} + 1 + \frac{q^2-1}{2g\sqrt{q}}\right)x_1 - 1 - \frac{q^2-1}{2g\sqrt{q}} = 0. \tag{14}$$

represents in the plane $\langle x_1, x_2\rangle$ a hyperbola $H^{q,g}$ that passes through the point $(-1,1)$. For $g \geq g_3 = \frac{\sqrt{q}(q-1)}{\sqrt{2}}$, the hyperbola $H^{q,g}$ intersects the parabola at least at three points. Hence we can have the following two configurations for the region of the plane which corresponds to the system (13):



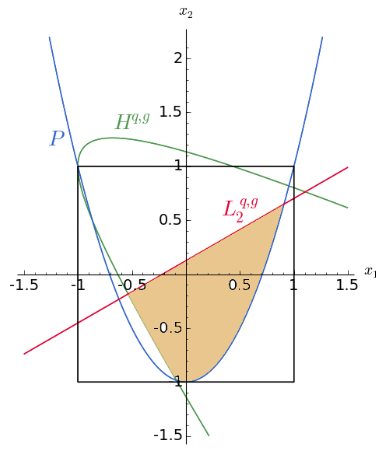


TABLE 4. The projection of $\mathcal{C}_3 \cap \mathcal{W}_3 \cap \mathcal{H}_3^{q,g}$ on the plane $< x_1, x_2 >$ respectively for $g < g_3$ and $g > g_3$.

We remark that for $g \geq g_3$ we have a better lower bound for x_2 as a function of x_1 (compared to the bound (11)), which is given by the smallest solution of the quadratic equation (14) in x_2:

$$x_2 \geq -x_1 - \sqrt{\frac{1}{q}x_1^2 + \left(\frac{1}{q} + 1 + \frac{q^2-1}{2g\sqrt{q}}\right)x_1 + 1 + \frac{q^2-1}{2g\sqrt{q}}}$$

Thus, by (9), we get a new upper bound for $B_2(X)$, in function of q, of g and of x_1:

$$B_2(X) \leq g\sqrt{q}(1+\sqrt{q})x_1 + gq\sqrt{\frac{1}{q}x_1^2 + \left(\frac{1}{q} + 1 + \frac{q^2-1}{2g\sqrt{q}}\right)x_1 + 1 + \frac{q^2-1}{2g\sqrt{q}}} + \frac{q^2-q}{2}. \tag{15}$$

Using equation (3) for x_1 in (15), we get a new upper bound for $B_2(X)$ as a function of q, g and $\sharp X(\mathbb{F}_q)$:

Proposition 3.3. *Let X be a smooth curve of genus $g \geq \frac{\sqrt{q}(q-1)}{\sqrt{2}}$ over $\mathbb{F}_q$. We have:*

$$B_2(X) \leq \sqrt{1/4\,(\sharp X(\mathbb{F}_q))^2 + \alpha(q,g)\sharp X(\mathbb{F}_q) + \beta(q,g)} \\ -\frac{(1+\sqrt{q})}{2}\sharp X(\mathbb{F}_q) + \frac{q^2+1+\sqrt{q}(q+1)}{2}, \tag{16}$$

where

$$\begin{cases} \alpha(q,g) = -\frac{1}{4}((2q\sqrt{q}+2\sqrt{q})g + q^3 + q + 2) \\ \beta(q,g) = \frac{1}{4}(4q^2g^2 + 2\sqrt{q}(q^3+q^2+q+1)g + q^4 + q^3 + q + 1). \end{cases}$$

As before, if we set

$$M'''(q,g) := \sqrt{1/4\,(N_q(g))^2 + \alpha(q,g)N_q(g)) + \beta(q,g)} \\ -\frac{(1+\sqrt{q})}{2}N_q(g)) + \frac{q^2+1+\sqrt{q}(q+1)}{2}, \tag{17}$$

where $\alpha(q,g)$ and $\beta(q,g)$ are defined as in Proposition 3.3, we have

$$B_2(\mathcal{X}_q(g)) \leq M'''(q,g).$$

By (2), we get the following proposition:

Proposition 3.4. *Let us assume that* $g \geq \frac{\sqrt{q}(q-1)}{\sqrt{2}}$.

If $\pi > g + M_q'''(g)$*, then no* δ*-optimal curves defined over* $\mathbb{F}_q$ *of geometric genus* g *and arithmetic genus* π *exist.*

In the following table, using the quantity $M'''(q,g)$, we give upper bounds for $B_2(\mathcal{X}_q(g))$. As $M'''(q,g)$ only makes sense when $g \geq \frac{\sqrt{q}(q-1)}{\sqrt{2}}$, some boxes of the table have been left empty.

q \ g	2	3	4	5	6
2	0	0	1	1	1
3		2	1	2	3
2^2				4	1

TABLE 5. Third-order upper bounds for $B_2(\mathcal{X}_q(g))$ given by $M'''(q,g)$.

Using Proposition 3.1 and Proposition 3.3, we can sum up Table 3 and 5 in the following one:

q \ g	2	3	4	5	6
2	0	0	1	1	1
3	3	2	1	2	3
2^2	5	0	4	4	1

TABLE 6. Upper bounds for $B_2(\mathcal{X}_q(g))$.

3.2. **Lower bound for** $B_2(X)$**.** In a similar way, we can look for lower bounds for $B_2(X)$. From the Weil bounds related to (4), we have $\sharp X(\mathbb{F}_{q^2}) \geq q^2+1-2gq$ and $\sharp X(\mathbb{F}_q) \leq q+1+2g\sqrt{q}$ so that

$$B_2(X) \geq \frac{q^2-q}{2} - g(q+\sqrt{q}). \tag{18}$$

It is easy to show that the right-hand side of (18) is positive if and only if $g < g_2 = \frac{\sqrt{q}(\sqrt{q}-1)}{2}$.

We can consider inequality (18) as a lower bound for $B_2(X)$ at the first order, as it is a direct consequence of the Weil bounds. Geometrically, it is also clear that we will not obtain better lower bounds at the second or at the third order. Indeed, looking at the graphics in Table 2 and Table 4, we remark that, in some cases and for some values of x_1, a better upper bound for x_2 is given by the line $L_2^{q,g}$. But we have seen in Remark 2.1 that if the pair (x_1, x_2) is on the line $L_2^{q,g}$, then $\sharp X(\mathbb{F}_q) = \sharp X(\mathbb{F}_{q^2})$, which means $B_2(X) = 0$.

For $g < g_2$, the inequality (18) implies the following lower bounds for $B_2(\mathcal{X}_q(g))$:

q \ g	2	3	4	5
7	2			
2^3	7			
3^2	12			
11	27	13		
13	45	29	12	
2^4	80	60	40	20

TABLE 7. Lower bounds for $B_2(\mathcal{X}_q(g))$.

Hence we get from the equivalence (2) and the inequality (18) the following proposition:

Proposition 3.5. *Let* $g < \frac{\sqrt{q}(\sqrt{q}-1)}{2}$.

If $g \leq \pi \leq g + \frac{q^2-q}{2} - g(q+\sqrt{q})$, *then there exists a* δ*-optimal curve defined over* $\mathbb{F}_q$ *of geometric genus* g *and arithmetic genus* π.

3.3. **Some exact values for** $N_q(g,\pi)$**.** We can use the previous results to provide some exact values of $N_q(g,\pi)$ for specific triples (q,g,π).

Proposition 3.6. *Let* q *be a power of a prime number* p. *We have:*

1. $N_q(0,\pi) = q+1+\pi$ *if and only if* $0 \leq \pi \leq \frac{q^2-q}{2}$.
2. *If* p *does not divide* $[2\sqrt{q}]$, *or* q *is a square, or* $q = p$, *then*
   $N_q(1,\pi) = q+[2\sqrt{q}]+\pi$ *if and only if* $1 \leq \pi \leq 1+\frac{q^2+q-[2\sqrt{q}]([2\sqrt{q}]+1)}{2}$.
   Otherwise,
   $N_q(1,\pi) = q+[2\sqrt{q}]+\pi-1$ *if and only if* $1 \leq \pi \leq 1+\frac{q^2+q+[2\sqrt{q}](1-[2\sqrt{q}])}{2}$.
3. *If* $g < \frac{\sqrt{q}(\sqrt{q}-1)}{2}$ *and* $g \leq \pi \leq \frac{q^2-q}{2} - g(q+\sqrt{q}-1)$ *then*
   $N_q(g,\pi) = N_q(g) + \pi - g$.
4. $N_2(2,3) = 6$.
5. $N_2(3,4) = 7$.
6. $N_{2^2}(4,5) = 14$.

Proof. Items *(1)* and *(2)* are Corollary 5.4 and Corollary 5.5 in [2]. Item *(3)* is given by Proposition 3.5.

We have that $N_2(2,3) \geq N_2(2) = 6$ and $B_2(\mathcal{X}_2(2)) = 0$, by Table 6. Hence *(4)* follows from Proposition 3.4 which says that $N_2(2,3) < N_2(2)+1$. Items *(5)* and *(6)* can be proven in a similar fashion.

□

Remark 3.7. Using the construction given in Section 3 of [2], we can easily show that $N_q(g,\pi+1) \geq N_q(g,\pi)$. This fact implies, for instance, that we have also $N_q\left(0, \frac{q^2-q}{2}+1\right) = q+1+\frac{q^2-q}{2}$.

4. Genera spectrum of maximal curves

Let X be a curve defined over $\mathbb{F}_q$ of geometric genus g and arithmetic genus π. We recall that X is a maximal curve if it attains bound (1), i.e

$$\sharp X(\mathbb{F}_q) = q+1+g[2\sqrt{q}]+\pi-g.$$

This definition extends the classical definition of a smooth maximal curve.

An easy consequence of Proposition 5.2 in [2] is that if X is a maximal curve, then its normalization $\tilde{X}$ is a smooth maximal curve. Moreover, the zeta function of a maximal curve X is given by (see Prop. 5.8 in [2]):

$$Z_X(T) = Z_{\tilde{X}}(T)(1+T)^{\pi-g} = \frac{(qT^2+[2\sqrt{q}]T+1)^g(1+T)^{\pi-g}}{(1-T)(1-qT)}.$$

We have seen in the previous section that, for π large enough compared to g, no maximal curve of geometric genus g and arithmetic genus π exist.

Hence, a related question concerns the genera spectrum of maximal curves defined over $\mathbb{F}_q$, i.e. the set of pairs (g, π), with $g, \pi \in \mathbb{N}$ and $g \leq \pi$, for which there exists a maximal curve over $\mathbb{F}_q$ of geometric genus g and arithmetic genus π:

$$\Gamma_q := \{(g, \pi) \in \mathbb{N} \times \mathbb{N} : \text{there exists a maximal curve defined over } \mathbb{F}_q \text{ of geometric genus } g \text{ and arithmetic genus } \pi\}.$$

The analogous question in the smooth case has been extensively studied in the case where q is a square. For q square, Ihara proved that if X is a maximal smooth curve defined over $\mathbb{F}_q$ of genus g, then $g \leq \frac{\sqrt{q}(\sqrt{q}-1)}{2}$ (see [11]) and Rück and Stichtenoth showed that g attains this upper bound if and only if X is $\mathbb{F}_q$-isomorphic to the Hermitian curve (see [14]). Moreover, Fuhrmann and Garcia proved that the genus g of maximal smooth curves defined over $\mathbb{F}_q$ satisfies (see [8])

$$\text{either} \qquad g \leq \left\lfloor \frac{(\sqrt{q}-1)^2}{4} \right\rfloor, \qquad \text{or} \qquad g = \frac{\sqrt{q}(\sqrt{q}-1)}{2}. \tag{19}$$

This fact corresponds to the so-called *first gap* in the spectrum genera of $\mathbb{F}_q$-maximal smooth curves. For q odd, Fuhrmann, Garcia and Torres showed that $g = \frac{(\sqrt{q}-1)^2}{4}$ occurs if and only if X is $\mathbb{F}_q$-isomorphic to the non-singular model of the plane curve of equation $y^{\sqrt{q}} + y = x^{\frac{\sqrt{q}+1}{2}}$ (see [7]). For q even, Abdón and Torres established a similar result in [1] under an extra condition that X has a particular Weierstrass point. In this case, $g = \frac{\sqrt{q}(\sqrt{q}-2)}{4}$ if and only if X is $\mathbb{F}_q$-isomorphic to the non-singular model of the plane curve of equation $y^{\sqrt{q}/2} + \cdots + y^2 + y = x^{(\sqrt{q}+1)}$.

Korchmáros and Torres improved (19) in [12]:

$$\text{either } g \leq \left\lfloor \frac{q-\sqrt{q}+4}{6} \right\rfloor, \text{ or } g = \left\lfloor \frac{(\sqrt{q}-1)^2}{4} \right\rfloor, \text{ or } g = \frac{\sqrt{q}(\sqrt{q}-1)}{2}. \tag{20}$$

Hence the *second gap* in the spectrum genera of $\mathbb{F}_q$-maximal smooth curves is also known. In the same paper, non-singular $\mathbb{F}_q$-models of genus $\left\lfloor \frac{q-\sqrt{q}+4}{6} \right\rfloor$ are provided.

Let us now consider maximal curves, possibly with singularities. We assume q to be a square and we want to study the discrete set Γ_q.

Let X be a maximal curve defined over $\mathbb{F}_q$ of geometric genus g and arithmetic genus π. As remarked above, the normalization $\tilde{X}$ of X is a maximal smooth curve, hence g satisfies (20). Moreover, g and π satisfy the following inequality:

Proposition 4.1. *Let q be a square. There exists a maximal curve defined over $\mathbb{F}_q$ of geometric genus g and arithmetic genus π if and only if $N_q(g) =$*

$q+1+2g\sqrt{q}$ *and*

$$g \le \pi \le g + \frac{q^2+(2g-1)q-2g\sqrt{q}(2\sqrt{q}+1)}{2}. \tag{21}$$

Proof. The proposition follows directly from the equivalence (2), from the fact that a maximal curve has a maximal normalization and that the number of closed points of degree 2 on a smooth maximal curve of genus g over $\mathbb{F}_q$ is given by (see Prop. 5.8 of [2]): $\frac{q^2+(2g-1)q-2g\sqrt{q}(2\sqrt{q}+1)}{2}$. □

Remark 4.2. The right-hand side of (21), which can be written as

$$(-q-\sqrt{q}+1)g + \frac{q^2-q}{2},$$

is a linearly decreasing with respect to g. Hence it attains its maximum value for $g=0$ (this also means that the number of closed points of degree 2 on a maximal smooth curve is a decreasing function of the genus). So we also get a bound for the arithmetic genus π in terms of the cardinality of the finite field:

$$\pi \le \frac{q(q-1)}{2}. \tag{22}$$

Geometrically, we have shown that the set Γ_q is contained in the triangle (OAB) (see Figure 1) of the plane $\langle g,\pi\rangle$ delimited by the lines $g=0$, $\pi=(-q-\sqrt{q}+1)g+\frac{q^2-q}{2}$ and $g=\pi$.

We observe that maximal curves over $\mathbb{F}_q$ with geometric genus $g=\frac{\sqrt{q}(\sqrt{q}-1)}{2}$ are necessarily smooth and thus isomorphic to the Hermitian curve.

Furthermore, the bound (22) is sharp. Indeed the singular plane rational curve provided in [9] is an example of a maximal curve defined over $\mathbb{F}_q$ with arithmetic genus $\pi=\frac{q(q-1)}{2}$.

Hence, using Proposition 3.6, the inequalities (20), Proposition 4.1 and Remark 4.2, we can state the following theorem:

Theorem 4.3. *Let q be a square and X be a maximal curve defined over $\mathbb{F}_q$ with geometric genus g and arithmetic genus π.*

If we set $g' := \frac{\sqrt{q}(\sqrt{q}-1)}{2}$, $g'' := \left\lfloor \frac{(\sqrt{q}-1)^2}{4} \right\rfloor$ and $g''' := \left\lfloor \frac{q-\sqrt{q}+4}{6} \right\rfloor$, then we have:

1. *$0 \le g \le g'$ and $g \le \pi \le \frac{q(q-1)}{2}$ and also $\pi \le g+\frac{q^2+(2g-1)q-2g\sqrt{q}(2\sqrt{q}+1)}{2}$. In other words Γ_q is contained in the set of integral points inside the triangle (OAB) of the following figure.*
2. *The point $B=(g',g')$ belongs to Γ_q and the following set of points $\left\{(0,\pi)\,,\ \text{with } 0 \le \pi \le \frac{q^2-q}{2}\right\}$ is contained in Γ_q.*
3. *If $g \ne g'$ then $g \le g''$ and the following set of points defined by $\left\{(g'',\pi)\,,\ \text{with } g'' \le \pi \le (-q-\sqrt{q}+1)g'' + \frac{q^2-q}{2}\right\}$ is contained in Γ_q.*

(4) *If $g \neq g'$ and $g \neq g''$, then $g \leq g'''$ and the following set of points* $\left\{(g''', \pi)\,,\ \text{with } g''' \leq \pi \leq (-q - \sqrt{q} + 1)g''' + \frac{q^2-q}{2}\right\}$ *is contained in* Γ_q.

We can illustrate Theorem 4.3 with the following figure (in which the aspect ratio has been set to 0.025 for readability):

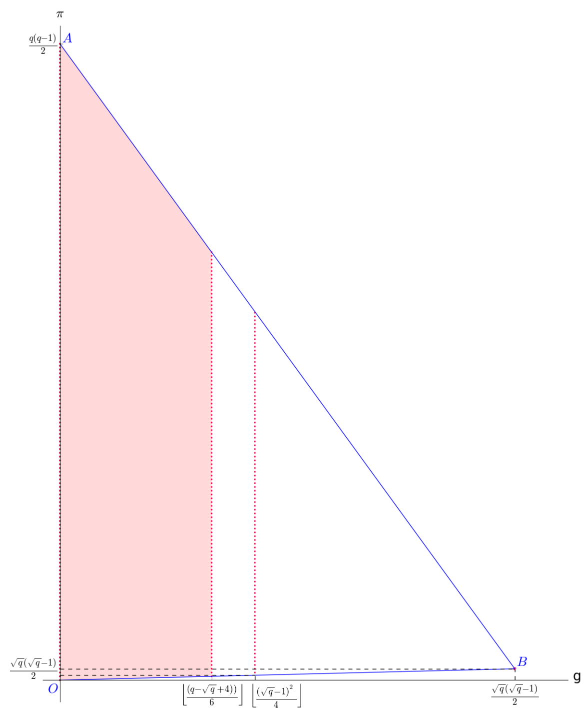


FIGURE 1. The set Γ_q is contained in the set of integral points inside the triangle (OAB). The dots correspond to the pairs (g, π) that we have proved to be in Γ_q. The rest of the set Γ_q has to be contained in the colored trapezoid.

We conclude the paper by considerations on coverings of singular curves. If $f : Y \to X$ is a surjective morphism of smooth curves defined over $\mathbb{F}_q$ and if Y is maximal then X is also maximal. This result is due to Serre (see [13]). We prove here that the result still holds without the smoothness assumption on the curves but with the assumption that the morphism is flat. Remark that the divisibility of the numerators of the zeta functions in a flat covering proved in [5] for possibly singular curves and in [6] for possibly singular varieties does not yield the result.

Theorem 4.4. *Let $f : Y \to X$ be a finite flat morphism between two curves defined over $\mathbb{F}_q$. If Y is maximal then X is maximal.*

Proof. Let us denote by g_X and π_X (respectively g_Y and π_Y) the geometric genus and the arithmetic genus of X (respectively of Y). As Y is maximal, we have

$$\sharp Y(\mathbb{F}_q) = q + 1 + g_Y[2\sqrt{q}] + \pi_Y - g_Y.$$

From Remark 4.1 of [3] we know that

$$|\sharp Y(\mathbb{F}_q) - \sharp X(\mathbb{F}_q)| \leq (\pi_Y - g_Y) - (\pi_X - g_X) + (g_Y - g_X)[2\sqrt{q}].$$

So we obtain:

$$\begin{aligned}\sharp X(\mathbb{F}_q) &\geq \sharp Y(\mathbb{F}_q) - (\pi_Y - g_Y) + (\pi_X - g_X) - (g_Y - g_X)[2\sqrt{q}] \\ &= q + 1 + g_X[2\sqrt{q}] + \pi_X - g_X.\end{aligned}$$

Hence X is also maximal. □

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(Yves Aubry) Institut de Mathématiques de Toulon, Université de Toulon, France, and Institut de Mathématiques de Marseille, CNRS-UMR 7373, Aix-Marseille Université, France
E-mail address: yves.aubry@univ-tln.fr

(Annamaria Iezzi) Institut de Mathématiques de Marseille, CNRS-UMR 7373, Aix-Marseille Université, France
E-mail address: annamaria.iezzi@univ-amu.fr