RATIONAL PLANE CURVES

Reference : Section 1.2 "Rational curves" in "Basic Algebraic Geometry 1", Shafarevich.

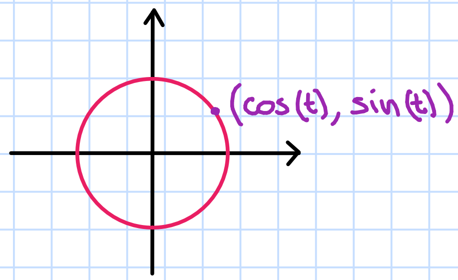
INTRODUCTION

Roughly speaking, rational curves are curves for which it is possible to find a "parametrization" given by rational functions.

We are already familiar with the concept of parametrization, i.e. the process of finding parametric equations of a curve defined by an implicit equation.

e.g: A parametrization of the unit circle $\mathcal{C}: x^2+y^2=1$ in $\mathbb{A}^2(\mathbb{R})$ is given by the mapping:

$$\begin{aligned} \varphi: \mathbb{R} &\longrightarrow \mathbb{A}^2(\mathbb{R}) \\ t &\longmapsto (\cos(t), \sin(t)) \end{aligned}$$



The parametrization φ is such that for every point P of the circle there exists (at least) a value t_0 such that $\varphi(t_0)=P$.

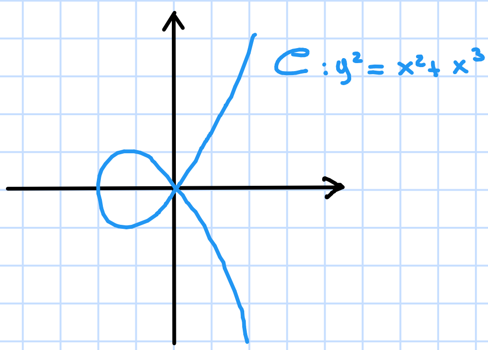
In algebraic geometry we only consider parametrizations given by rational functions.

For the rest of this lecture we will assume K to be an algebraically closed field. Nevertheless, in order to keep a geometrical intuition, we will draw some examples of curves in the real plane.

A CLASSICAL EXAMPLE

Let us consider the curve given by the polynomial:

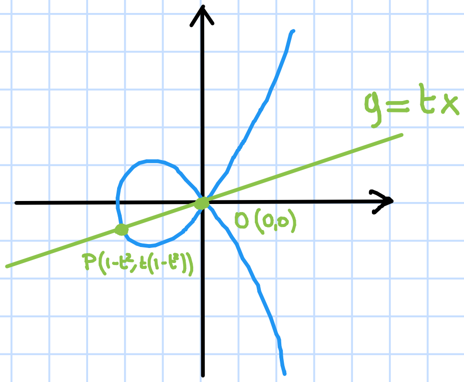
$$f(x,y) = y^2 - x^2 - x^3 \longrightarrow \mathcal{C}: y^2 = x^2 + x^3$$



For each $t \in K$, the line $y = tx$ (with slope t and passing through the origin $(0,0)$) intersects the curve $\mathcal{C}$ in exactly two points:

$$\begin{cases} y^2 = x^2 + x^3 \\ y = tx \end{cases} \Longrightarrow t^2x^2 = x^2 + x^3 \Longrightarrow x^2(t^2 - 1 - x) = 0 \quad \begin{matrix} \nearrow x = 0 \\ \searrow x = t^2 - 1 \end{matrix}$$

$\Longrightarrow$ $O(0,0)$ and $P(t^2-1, t(t^2-1))$



Then, for every $t \in K$, the point $(t^2-1, t(t^2-1))$ belongs to $\mathcal{C}$. Indeed, note that:

$$f(t^2-1, t(t^2-1)) \equiv 0 \quad , \text{ as an identity of } t.$$

Thus we have a map:

$$\begin{aligned} \gamma : K &\longrightarrow \mathcal{C} \\ t &\longmapsto (\varphi(t), \psi(t)) = (t^2-1, t(t^2-1)), \end{aligned}$$

where $\varphi(t) = t^2-1$ and $\psi(t) = t(t^2-1)$ are rational functions of t (in this case they are polynomials) and

$$f(\varphi(t), \psi(t)) \equiv 0.$$

Vice versa, each point $P(x,y) \in \mathcal{C}$, $P \neq (0,0)$ is sent on the slope of the line passing through $(0,0)$ and P, i.e. $\frac{y}{x}$.

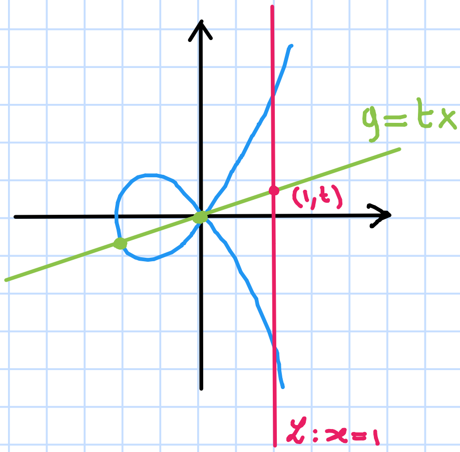
In other terms, we have the following map:

$$\begin{aligned} \delta : \mathcal{C} \setminus \{0,0\} &\longrightarrow K \\ (x,y) &\longmapsto \chi(x,y) = \frac{y}{x} \end{aligned},$$

where $\chi(x,y) = \frac{y}{x}$ is a rational function of the coordinates of the point (x,y).

We can also interpret the map δ as a projection of the curve $\mathcal{C}$ on the line $\mathcal{L} : x = 1$:

every point $P(x,y) \in \mathcal{C}$, $P \neq (0,0)$, is sent on the point $(1,t) \in \mathcal{L}$, where t is the slope of the line passing through $(0,0)$ and P.



Note that the maps γ and δ are not bijections :

- γ is not injective : $\gamma(1) = \gamma(-1) = (0,0)$
- δ is not surjective : $1, -1 \notin \mathrm{Im}(\delta)$

Nevertheless we can have a bijection, provided that we remove a finite set of points from both $\mathcal{C}$ and K :

$$\mathcal{C} \setminus \{0,0\} \longleftrightarrow K \setminus \{1,-1\}$$

$$(x,y) \longmapsto \chi(x,y) = \frac{y}{x}$$

$$(\varphi(t), \psi(t)) = (t^2-1, t(t^2-1)) \longleftarrow\!\!| \; t$$

We will see that γ and δ are examples of "rational maps" which are inverses of each other, and we will say that $\mathcal{C}$ is "birationally equivalent" to a line.

The curve $\mathcal{C}$ is an example of rational curve.

FORMAL DEFINITION

Recall : We denote by $K(t)$ the field of rational functions of t :

$$K(t) = \left\{ \frac{f(t)}{g(t)} \, , \; f, g \in K[t], \; g \neq 0 \right\}.$$

Note that $K(t)$ is the field of fractions of $K[t]$, i.e. the smallest field that contains $K[t]$.

Def: Let $\mathcal{C}$ be an irreducible algebraic curve defined by $f(x,y) \in K[x,y]$:

$$\mathcal{C}: f(x,y) = 0.$$

$\mathcal{C}$ is rational if there exist two rational functions $\varphi(t), \psi(t) \in K(t)$, at least one non constant, such that

$$f(\varphi(t), \psi(t)) \equiv 0, \text{ as an identity of } t.$$

In this case the map:

$$\begin{aligned} \gamma: K &\longrightarrow \mathcal{C} \\ t &\longmapsto (\varphi(t), \psi(t)) \end{aligned}$$

is called a parametrization of $\mathcal{C}$.

Remarks:

- The map γ is defined for all $t \in K$, except possibly finitely many values of t.

  Indeed, let $\{\alpha_1, \dots, \alpha_n\} \subseteq K$ be the finite set of values that vanish the denominators of either $\varphi(t)$ or $\psi(t)$.

  Then, for every $t_0 \in K \setminus \{\alpha_1, \dots, \alpha_n\}$, $\gamma(t_0) = (\varphi(t_0), \psi(t_0))$ is a point on $\mathcal{C}$.

- Let $K_0 \subseteq K$ be a subfield of K (e.g. $\mathbb{Q} \subseteq \mathbb{C}$, $\mathbb{F}_q \subseteq \overline{\mathbb{F}_q}$, etc.)

  If $\varphi(t), \psi(t) \in K_0(t)$ (i.e. the coefficients of the rational functions φ and ψ belong to K_0) and $t_0 \in K_0$, then the point $(\varphi(t_0), \psi(t_0))$ has coordinates in K_0, i.e. it is K_0-rational.

We will see that, if $\mathcal{C}$ is a rational curve, then there exists a suitable choice of the parametrization $(\varphi(t), \psi(t))$, such that the map

$$\begin{aligned} K &\longrightarrow \mathcal{C} \\ t &\longmapsto (\varphi(t), \psi(t)) \end{aligned}$$

is a bijection between the set of values of t and the points on the curve, provided that we exclude certain finite sets from both the set of values of t and the points on the curve.

Thus, conversely, we will see that the parameter t can be expressed as a rational function $t = \chi(x,y)$ of the coordinates x and y:

$$\begin{aligned} K &\longrightarrow \mathcal{C} \\ t &\longmapsto (\varphi(t), \psi(t)) \end{aligned} \qquad \qquad \begin{aligned} \mathcal{C} &\longrightarrow K \\ (x,y) &\longmapsto \chi(x,y) \end{aligned}$$

RATIONAL FUNCTIONS

WHICH CURVES ARE RATIONAL?

Not all algebraic curves are rational. Nevertheless, in the plane all algebraic curves of degree 1 (lines) and of degree 2 (conics) are rational.

* CURVES OF DEGREE 1 (LINES)

$f(x,y) = ax+by+c \in k[x,y]$, $\quad \mathcal{C}: f(x,y)=0$

Then the map

$$\begin{aligned} k &\longrightarrow \mathcal{C} \\ t &\longmapsto \left(t, \frac{-at-c}{b}\right) \end{aligned}$$

is a parametrization of $\mathcal{C}$ which is also a bijection between k and $\mathcal{C}$.

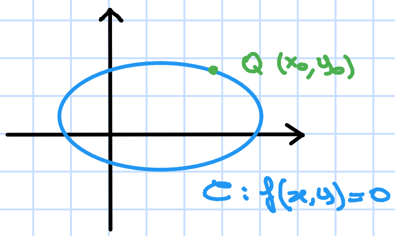
Hence every line is a rational curve.

* CURVES OF DEGREE 2 (CONICS)

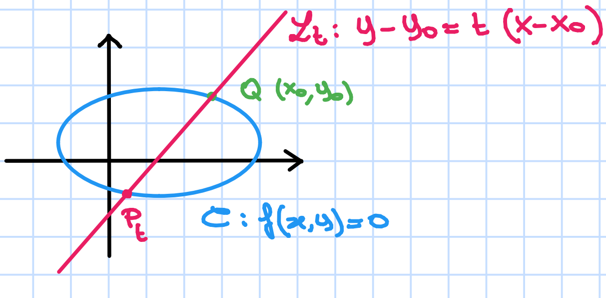
Let $\mathcal{C}$ be an irreducible conic, defined by the polynomial $f(x,y) \in k[x,y]$ of degree 2.

Consider the following geometrical construction:

1) Choose a point $Q(x_0,y_0) \in \mathcal{C}$.



2) Draw the line $\mathcal{L}_t$ with slope t which passes through Q.



3) For all t, $\mathcal{L}_t$ intersects $\mathcal{C}$ in two points: $Q(x_0,y_0)$ and $P_t(x_t,y_t)$ (note that if $\mathcal{L}_t$ is tangent to $\mathcal{C}$, then $P_t = Q$).

Find the coordinates of P_t:

$$\begin{cases} f(x,y)=0 \\ y-y_0=t(x-x_0) \end{cases} \implies \underbrace{f(x,\ y_0+t(x-x_0))=0}_{\substack{\downarrow \\ \text{This is a polynomial equation of} \\ \text{degree 2 in } x \text{ for which } x_0,\ x_t \in K \\ \text{are the two solutions (recall that} \\ K \text{ is algebraically closed...).}}} \quad (*)$$

We can rewrite $(*)$ in the following way:

$$A(t)x^2+B(t)x+C(t)=0, \quad \text{where } A(t),B(t),C(t)\in K[t].$$

Hence we have $x_t+x_0=-\dfrac{B(t)}{A(t)}\in K(t)$, i.e:

$$x_t=-x_0-\frac{B(t)}{A(t)}\in K(t).$$

Then $y_t=y_0+t(x_t-x_0)$, i.e.

$$y_t=y_0+t\left(-2x_0-\frac{B(t)}{A(t)}\right)\in K(t).$$

We get that $\mathcal{C}$ is a rational curve and a parametrization is given by:

$$\begin{aligned} K &\longrightarrow \mathcal{C} \\ t &\longmapsto \Bigg(\underbrace{-x_0-\frac{B(t)}{A(t)}}_{\varphi(t)},\ \underbrace{y_0-2tx_0-t\frac{B(t)}{A(t)}}_{\psi(t)}\Bigg) \end{aligned}$$

<u>Remark</u>: In constructing the previous parametrization we have used a point (x_0,y_0) on the curve $\mathcal{C}$.

Let $K_0\subseteq K$ be a subfield of K. If $(x_0,y_0)\in \mathbb{A}^2(K_0)$ and $f(x,y)\in K_0[x,y]$ then $\varphi(t),\psi(t)\in K_0(t)$.

This implies that $\forall\, t_0\in K_0$, the point $(\varphi(t_0),\psi(t_0))\in \mathbb{A}^2(K_0)$

Moreover, it is easy to show that for every point (x,y) on $\mathcal{C}$ with coordinates in K_0 (except possibley finitely many) there is $t_0\in K_0$ such that $(x,y)=(\varphi(t_0),\psi(t_0))$.

Thus, for example, the previous parametrization gives as the general form for the solution in K_0 of an indeterminate equation of degree 2 if we know just one solution.

Hence the problem of finding the solutions in K_0 of a polynomial equation of degree 2:

$$f(x,y)=0, \qquad f(x,y) \in K_0[x,y]$$

boils down in finding a solution $(x_0,y_0) \in K_0^2$.

The question of whether this solution exists is delicate. For $K_0=\mathbb{Q}$, it is solved by Legendre's theorem.

<u>Remark</u>: It is not difficult to show that it is always possible to reduce the equation of a conic:

$$f(x,y)= ax^2+by^2+cxy+dx+ey+f=0, \quad a,b,c,d,e,f\in\mathbb{Q} \quad (+)$$

to an equation of the form:

$$a'x^2+b'y^2+c'z^2=0, \quad a',b',c'\in\mathbb{Z}, \quad a'b'c'\neq 0 \text{ squarefree} \quad (++)$$

such that

$(+)$ has a rational solution (in $\mathbb{Q}$) $\iff$ $(++)$ has a nontrivial integer solution (in $\mathbb{Z}$).

<u>Legendre's theorem</u>: Let a,b,c be non zero integers such that abc is squarefree. Then

$$ax^2+by^2+cz^2=0$$

has a nontrivial integer solution if and only if all the following conditions are satisfied:

- a,b,c do not all have the same sign;
- $-ab$ is a square modulo c;
- $-bc$ is a square modulo a;
- $-ac$ is a square modulo b.