AFFINE ALGEBRAIC SETS

Reference: Sections 1.2, 1.3, 1.4 "Algebraic curves", Fulton

From now on we will always assume K algebraically closed, unless otherwise specified.

So far we worked in the affine plane $\mathbb{A}^2(K)$ where we need only one polynomial $f(x,y) \in K[x,y]$ in order to describe a curve.

Nevertheless, (affine) algebraic curves can live also in affine spaces of higher dimension.
In this case we will see that one polynomial is not enough for describing a curve. For instance, in the affine space of dimension three, we can obtain a line as the intersection of two planes, and a conic as the intersection of the surface of a cone with a plane.
We can also have curves that do not lie on a plane.

ALGEBRAIC SETS

We call $\mathbb{A}^n(K)$ the affine n-space over K:

$$\mathbb{A}^n(K) = \{(a_1,\dots,a_n) : a_i \in K\}.$$

As a set, $\mathbb{A}^n(K)$ is simply the cartesian product of K with itself n times. It has also the structure of a vector space of dimension n.

Each element $(a_1,\dots,a_n) \in \mathbb{A}^n(K)$ is called a point.

For $n=1$, $\mathbb{A}^1(K)$ is called the affine line; for $n=2$, $\mathbb{A}^2(K)$ is the affine plane.

Now, since in $\mathbb{A}^n(K)$ each point has n coordinates, we need to consider polynomials in n variables.

Let $F \in K[x_1,\dots,x_n]$ be a polynomial in n variables with coefficients in K. A point $P = (a_1,\dots,a_n) \in \mathbb{A}^n(K)$ is called a zero of F if

$$F(P) := F(a_1,\dots,a_n) = 0$$

We consider the set of zeros of F, denoted by:

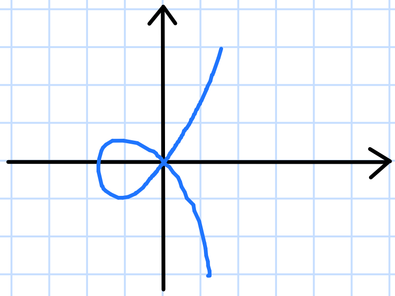
$$V(F) = \{P \in \mathbb{A}^n(K) : F(P) = 0\}$$

If F is nonconstant, $V(F)$ is called the hypersurface defined by F.

Note that:

- an hypersurface in $\mathbb{A}^2(K)$ is an affine plane algebraic curve.

  e.g.: $V(Y^2 - X^2 - X^3) = \{(x,y) \in \mathbb{A}^2(K) : y^2 = x^2 + x^3\}$



- if $\deg F = 1$ then $V(F)$ is called a hyperplane in $\mathbb{A}^n(K)$.

  An hyperplane in $\mathbb{A}^2(K)$ is a line.

- If $K = \mathbb{R}$,

  $V(X^2 + Y^2 + Z^2 - 1) \subseteq \mathbb{A}^3(\mathbb{R})$ is a sphere.

  $V(X^2 + Y^2 - Z^2) \subseteq \mathbb{A}^3(\mathbb{R})$ is a cone.

Now, let $S \subseteq K[X_1, \dots, X_n]$ be a subset of polynomials. Then we can consider the set of points in $\mathbb{A}^n(K)$ which are common zeros for all polynomials in S.

$$V(S) = \{P \in \mathbb{A}^n(K) : F(P) = 0 \ \forall\, F \in S\} = \bigcap_{F \in S} V(F)$$

If $S = \{F_1, \dots, F_r\}$ is a finite set, we write $V(F_1, \dots, F_r)$ instead of $V(\{F_1, \dots, F_r\})$.

Def: A subset $X \subseteq \mathbb{A}^n(K)$ is an affine algebraic set if

$$X = V(S)$$

for some $S \subseteq K[X_1, \dots, X_n]$.

Recall

Let R be a commutative ring (with unity 1).

Def: A nonempty subset $I \subseteq R$ is an ideal if $\forall\, x, y \in I$, $\forall\, r \in R$, $x + y \in I$, $rx \in I$.

Def: If $S \subseteq R$ is a subset of R, we denote by:

$$(S) := \{ r_1 s_1 + \dots + r_n s_n \; : \; n \in \mathbb{N}, \; r_i \in R, \; s_i \in S \}$$

the ideal generated by S. It is the smallest ideal containing S.

Def: If $I, J \subseteq R$ are ideals then:

$$I + J = \{ a + b : \; a \in I, \; b \in J \}$$

$$IJ = \{ a_1 b_1 + \dots + a_n b_n : \; a_i \in I, \; b_i \in J, \; n \in \mathbb{N} \}$$

are ideals, called respectively the sum and the product of I and J.

Properties

1) If $I = S$ then $V(S) = V(I)$

So every algebraic set $X = V(I)$, for some ideal $I \subseteq K[X_1, \dots, X_n]$

2) $I \subseteq J \Rightarrow V(I) \supseteq V(J)$

3) $V(0) = \mathbb{A}^n(K)$, $V(1) = \emptyset$, $V(X_1 - a_1, \dots, X_n - a_n) = (a_1, \dots, a_n)$

So $\mathbb{A}^n(K)$, $\emptyset$ and every point of $\mathbb{A}^n(K)$ are algebraic set.

4) If $\{ I_\alpha \}$ is any collection of ideals then

$$\cap_\alpha V(I_\alpha) = V(\cup_\alpha I_\alpha)$$

So any intersection of algebraic set is an algebraic set

In particular $V(I) \cap V(J) = V(I \cup J) = V(I + J)$

the union of two ideals is not in general an ideal. The sum $I+J$ is the ideal generated by the union $I \cup J$.

5) $V(I) \cup V(J) = V(IJ)$

So any union of finitely many algebraic sets is an algebraic set.

In particular, if $F, G \in K[X_1, \dots, X_n]$ then $V(F) \cup V(G) = V(FG)$.

REMARK : • Because of 3), 4), 5) $\mathbb{A}^n(K)$ is a topological space where the closed sets are all the algebraic sets in $\mathbb{A}^n(K)$.

This topology is called the ZARISKI TOPOLOGY on $\mathbb{A}^n(K)$.

• Because of 5, every finite set of $\mathbb{A}^n(K)$ is an algebraic set.

Proof

1) $I = (S) = \{ F_1 G_1 + \dots + F_n G_n : F_i \in S, \ G_i \in K[X_1, \dots, X_n] \}$

$\longrightarrow V(I) \subseteq V(S)$

$P \in V(I) \Rightarrow F(P) = 0 \ \forall F \in I$. But $S \subseteq I$, then $F(P) = 0 \ \forall F \in S \Rightarrow P \in V(S)$.

$\longrightarrow V(S) \subseteq V(I)$

if $P \in V(S) \Rightarrow F(P) = 0 \ \forall F \in S$. Now, if $F_i \in S$, $G_i \in K[X_1, \dots, X_n]$ we have:

$$\left(\sum_{i=1}^{n} F_i G_i \right)(P) = \sum_{i=1}^{n} F_i(P) G_i(P) \underset{\substack{\uparrow \\ F_i(P)=0}}{=} 0 \Rightarrow P \in V(I).$$

2) Trivial.

3) Trivial.

4) $\longrightarrow \bigcap_\alpha V(I_\alpha) \subseteq V\left(\bigcup_\alpha I_\alpha \right)$

$P \in \bigcap_\alpha V(I_\alpha) \Rightarrow P \in V(I_\alpha) \ \forall \alpha \Rightarrow$

$\Rightarrow F(P) = 0 \ \forall F \in I_\alpha, \forall \alpha \Rightarrow F(P) = 0 \ \forall F \in \bigcup_\alpha I_\alpha$

$\Rightarrow P \in V\left(\bigcup_\alpha I_\alpha \right)$.

$\longrightarrow V\left(\bigcup_\alpha I_\alpha \right) \subseteq \bigcap_\alpha V(I_\alpha)$

$P \in V\left(\bigcup_\alpha I_\alpha \right) \Rightarrow F(P) = 0 \ \forall F \in \bigcup_\alpha I_\alpha \Rightarrow$

$\Rightarrow \forall \alpha, F(P) = 0 \ \forall F \in I_\alpha \Rightarrow \forall \alpha \ P \in V(I_\alpha) \Rightarrow P \in \bigcap_\alpha V(I_\alpha)$.

5) $\longrightarrow V(I) \cup V(J) \subseteq V(IJ)$

$P \in V(I) \cup V(J) \Rightarrow F(P) = 0 \ \forall F \in I$, or $G(P) = 0 \ \forall G \in J$

Now, if $F \in IJ \Rightarrow F = F_1 G_1 + \dots + F_n G_n$, $F_i \in I$, $G_i \in J$,
$$\Rightarrow F(P) = \sum_{i=1}^{n} F_i(P) G_i(P) \underset{\substack{\uparrow \\ F_i(P)=0 \text{ or } G_i(P)=0}}{=} 0 \Rightarrow P \in V(IJ).$$

$\longrightarrow V(IJ) \subseteq V(I) \cup V(J)$

Let $P \in V(IJ) \Rightarrow \sum_{i=1}^{n} F_i(P) G_i(P) = 0$, $F_i \in I$, $G_i \in J$.

If $P \notin V(I) \Rightarrow \exists\, F \in I$ such that $F(P) \neq 0$. Since $P \in V(IJ)$ then $\forall G \in J$ $F(P)G(P) = 0 \Rightarrow \forall G \in J$, $G(P) = 0 \Rightarrow P \in V(J)$.

THE IDEAL OF A SET OF POINTS

In the previous section, we have associated to a set of polynomials a set of points of $\mathbb{A}^n(K)$.

Now, conversely, we will associate to a set of points of $\mathbb{A}^n(K)$ a set of polynomials of $K[X_1, \dots, X_n]$.

Let $X \subseteq \mathbb{A}^n(K)$. We consider the set of polynomials that vanish on X:

$$I(X) = \{F \in K[X_1, \dots, X_n] : F(P) = 0 \ \forall P \in X\}$$

We see that $I(X) \subseteq K[X_1, \dots, X_n]$ is an ideal of X.

Indeed if $F, G \in I(X) \Rightarrow F(P) + G(P) = 0 \ \forall P \in X \Rightarrow$
$\Rightarrow F + G \in I(X)$.
Moreover if $F \in I(X)$, $G \in K[X_1, \dots, X_n] \Rightarrow F(P)G(P) = 0 \ \forall P \in X$
$\Rightarrow FG \in I(X)$.

$I(X)$ is called the ideal of X.

Properties

1) $X \subseteq Y \Rightarrow I(X) \supseteq I(Y)$.

2) $I(\emptyset) = K[X_1, \dots, X_n]$, $I(\mathbb{A}^n(K)) = (0)$, $I(\{(a_1, \dots, a_n)\}) = (X_1 - a_1, \dots, X_n - a_n)$.

3) $S \subseteq I(V(S))$, $\forall S \subseteq K[X_1, \dots, X_n]$
$X \subseteq V(I(X))$, $\forall X \subseteq \mathbb{A}^n(K)$

4) $V(I(V(S))) = V(S)$, $\forall S \subseteq K[X_1, \dots, X_n]$
$I(V(I(X))) = I(X)$, $\forall X \subseteq \mathbb{A}^n(K)$.

In particular, if V is an algebraic set we have $V(I(V)) = V$, and if I is an ideal of a set of points $I(V(I)) = I$.

Remark : • The identity $I(\mathbb{A}^n(K)) = (0)$ is true since K is infinite, but it does not hold in general.

In order to prove it, consider a polynomial $F \in I(\mathbb{A}^n(K))$ and fix $a_1, \dots, a_{n-1} \in K$.

Then $F(a_1, \dots, a_{n-1}, X_n) = f(X_n) \in K[X_n]$ satisfies $f(a_n) = 0 \ \forall a_n \in K \Rightarrow f \equiv 0$. This implies that the degree of F in X_n is 0.
By repeating this for the other variables we get $F \equiv 0$.

• The inclusions in 3) can be proper:

→ if $S = (x^2) \subseteq K[X,Y]$ then:

$$I(V(x^2)) = I(\{(0,y) : y \in K\}) = (x) \supsetneq (x^2) = S.$$

→ if $X = \{(0,y),\ y \neq 0\}$ then:

$$V(I(X)) = V((x)) = \{(0,y)\ y \in K\} \supsetneq X.$$

Recall

Def: Let R be a commutative ring and $I \subseteq R$ an ideal. Then, the radical of I, denoted by $\mathrm{Rad}(I)$ or $\sqrt{I}$, is defined as:

$$\mathrm{Rad}(I) = \sqrt{I} := \{ r \in R \mid r^n \in I, \text{ for some } n > 0 \}.$$

Note that $\mathrm{Rad}(I)$ is an ideal that contains I: $I \subseteq \mathrm{Rad}(I)$.

We have also $\mathrm{Rad}(I) = \bigcap_{\substack{P \supseteq I \\ P \text{ prime}}} P$.

e.g.: • if $R = \mathbb{Z}$ and $I = 4\mathbb{Z}$, then

$$\mathrm{Rad}(I) = \{ a \in \mathbb{Z} \text{ s.t. } a^n \in 4\mathbb{Z},\ n > 0 \} = 2\mathbb{Z}$$

• if $R = K[x,y]$ and $I = (x^2 y^3)$, then

$$\mathrm{Rad}(I) = \{ f \in K[x,y] \text{ s.t. } f^n \in I,\ n > 0 \} = (xy)$$

Def: An ideal $I \subseteq R$ is said to be radical if

$$\mathrm{Rad}(I) = I.$$

e.g: $(x) \subseteq k[x,y]$ is radical since $\mathrm{Rad}(I) = (x)$.

Proposition: Let $X \subseteq \mathbb{A}^n(K)$. Then $I(X)$ is a radical ideal.

Proof:

We have only to show that $\mathrm{Rad}(I(X)) \subseteq I(X)$.

Let $F \in \mathrm{Rad}(I(X))$. Then $\exists\, n > 0$ such that $F^n \in I(X) \Rightarrow$

$\Rightarrow F^n(P) = 0 \quad \forall\, P \in X \Longrightarrow F(P) = 0 \;\forall\, P \in I(X) \Longrightarrow F \in I(X)$.

THE HILBERT BASIS THEOREM

Recall: A commutative ring R is Noetherian if every ideal of R is finitely generated.

e.g: Every PID ($\mathbb{Z}$, $k[x]$, etc.) and field is Noetherian.

Theorem (HILBERT BASIS THEOREM)

If R is a Noetherian ring, then $R[X]$ is Noetherian.

Corollary: If R is a Noetherian ring, then $R[X_1, \dots, X_n]$ is Noetherian.
In particular, if K is a field, then $K[X_1, \dots, X_n]$ is Noetherian.

Proposition: Every algebraic set is an intersection of a finite number of hypersurfaces.

Proof

If $X \subseteq \mathbb{A}^n(K)$ is an algebraic set, then $X = V(I)$ for some ideal $I \subseteq k[X_1, \dots, X_n]$.

Since $K[X_1, \dots, X_n]$ is Noetherian, I is finitely generated, i.e

$$I = (F_1, \dots, F_r), \quad F_i \in K[X_1, \dots, X_n].$$

Then $X = V(I) = V(F_1, \dots, F_r) = \bigcap_{i=1}^{r} V(F_i)$ is the intersection of the hypersurfaces described by $F_1, \dots, F_r$.