IRREDUCIBLE AFFINE ALGEBRAIC SETS

Reference: Sections 1.5, 1.7 "Algebraic curves", Fulton.

Some algebraic sets can be written as the union of "smaller" algebraic sets.

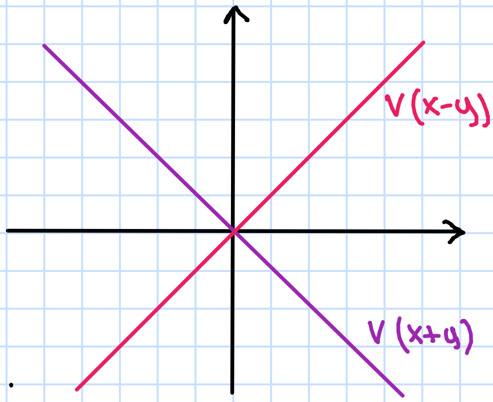
e.g: Let $V = V(x^2-y^2) \subseteq \mathbb{A}^2(K)$.

Since $x^2-y^2 = (x-y)(x+y)$, we have

$$V = V(x-y) \cup V(x+y),$$

with $V(x-y) \subsetneq V$ and $V(x+y) \subsetneq V$.

We call V a "reducible" algebraic set.



Def: An algebraic set $V \subseteq \mathbb{A}^n(K)$ is reducible if

$$V = V_1 \cup V_2,$$

where V_1, V_2 are algebraic sets in $\mathbb{A}^n(K)$, with $V_1 \subsetneq V$, $V_2 \subsetneq V$. Otherwise we say that V is irreducible.

e.g: Consider the algebraic set $V = V(x^2+y^2) \subseteq \mathbb{A}^2(\mathbb{C})$.

We have:

$$V = V(x+iy) \cup V(x-iy),$$

with $V(x+iy) \subsetneq V$ $\left((i,1) \in V \setminus V(x+iy)\right)$ and $V(x-iy) \subsetneq V$ $\left((-i,1) \in V \setminus V(x-iy)\right)$. So V is reducible.

We remark that

$$I(V) = I(V(x^2+y^2)) = (x^2+y^2) \subseteq \mathbb{C}[x,y]$$

is not a prime ideal, since $x^2+y^2 = (x+iy)(x-iy)$ is not a prime (= irreducible) element of $\mathbb{C}[x,y]$.

If now we consider $V(x^2+y^2)$ in the real plane $\mathbb{A}^2(\mathbb{R})$, we can not find two algebraic sets $V_1 \subsetneq V$, $V_2 \subsetneq V$ such that $V = V_1 \cup V_2$. Indeed the polynomial x^2+y^2 is irreducible in $\mathbb{R}[x,y]$.

Therefore $V(x^2+y^2)$ is irreducible in $\mathbb{A}^2(\mathbb{R})$.

Recall

Def: Let R be a (commutative) ring.

An ideal $I \subsetneq R$ is prime if for all $a, b \in R$ such that $ab \in I$ we have either $a \in I$ or $b \in I$.

e.g: If $R = \mathbb{Z}$, then $I = a\mathbb{Z}$ is a prime ideal if and only if a is a prime number or $I = (0)$.

If R is an UFD and $a \in R$, then

(a) is a prime ideal $\iff$ a is an irreducible ($=$ prime) element

e.g: If $R = K[x_1, \dots, x_n]$, then $I = (F(x_1, \dots, x_n))$ is prime if and only if $F(x_1, \dots, x_n)$ is an irreducible polynomial.

We have the following result:

Proposition: Let K be an arbitrary field.

An algebraic set V is irreducible if and only if $I(V)$ is a prime ideal.

Proof

$\Rightarrow$) By contrapositive, assume that $I(V)$ is not prime.

Then $\exists\ F, G \in K[x_1, \dots x_n]$ such that $FG \in I(V)$ with

① $F \notin I(V)$ and

② $G \notin I(V)$.

We have:

① $\Rightarrow \exists\ P \in V$ such that $F(P) \neq 0 \xRightarrow{P \in V \setminus V(F)} V_1 = V(F) \cap V \subsetneq V$

② $\Rightarrow \exists\ Q \in V$ such that $G(Q) \neq 0 \xRightarrow{Q \in V \setminus V(G)} V_2 = V(G) \cap V \subsetneq V$

and:

$V(FG) \supseteq V$

$$V_1 \cup V_2 = (V(F) \cap V) \cup (V(G) \cap V) = (V(F) \cup V(G)) \cap V = V(FG) \cap V = V.$$

Hence we obtain that V is reducible.

$\Leftarrow$) By contrapositive, assume that V is reducible. Then

$$V = V_1 \cup V_2, \quad V_1 \subsetneq V \text{ (1)} \quad \text{and} \quad V_2 \subsetneq V \text{ (2)}$$

$(1) \Rightarrow I(V_1) \supsetneq I(V) \Rightarrow \exists\, F \in I(V_1) \setminus I(V)$

$(2) \Rightarrow I(V_2) \supsetneq I(V) \Rightarrow \exists\, G \in I(V_2) \setminus I(V)$.

Moreover,

$$FG \in I(V_1) \cap I(V_2) = I(V_1 \cup V_2) = I(V),$$

which implies that $I(V)$ is not prime.

$\square$

Question: If $I \subsetneq K[x_1, \dots, x_n]$ is a prime ideal, then is $V(I)$ irreducible?

In other words, if I is prime, is $I(V(I))$ prime?

We will come back to this question in next class...

We will see now that any algebraic set is the union of a finite number of irreducible algebraic sets.

Proposition: Let V be an algebraic set in $\mathbb{A}^n(K)$. Then there are unique irreducible algebraic sets $V_1, \dots, V_m$ such that

$$V = V_1 \cup \dots \cup V_m$$

and $V_i \not\subseteq V_j$, $\forall\, i \neq j$

The union $V_1 \cup \dots \cup V_m$ is called the decomposition into irreducible components of V.

In order to prove the previous proposition, we will use the following fact from commutative algebra:

Fact: If $\mathcal{I} = \{I_\alpha\}_\alpha$ is a nonempty collection of ideals of a Noetherian ring R. Then $\mathcal{I}$ has a <u>maximal element</u>, i.e. $\exists\ I_{\alpha_0} \in \mathcal{I}$ such that $I_{\alpha_0} \not\subsetneq I_\alpha$ $\forall \alpha \neq \alpha_0$.

Now, let $\mathcal{S} = \{V_\alpha\}_\alpha$ be a nonempty collection of algebraic sets in $\mathbb{A}^n(k)$.

Then $\mathcal{I} = \{I(V_\alpha)\}_\alpha$ is a nonempty collection of ideals of $K[X_1, \dots, X_n]$ which, by the previous fact, has a maximal element $I(V_{\alpha_0})$. We have that V_{α_0} is a <u>minimal element</u> for $\mathcal{S}$.

Proof

• EXISTENCE OF A FINITE DECOMPOSITION

Let us consider the following set:

$$\mathcal{S} = \{\text{algebraic sets } V \subseteq \mathbb{A}^n(k) : V \text{ is not the union of a finite number of irreducible algebraic sets}\}$$

We want to show that $\mathcal{S} = \emptyset$.

By contrapositive, if $\mathcal{S} \neq \emptyset$, then there exists a minimal element $V \in \mathcal{S}$.

Note that V cannot be irreducible (otherwise V would not belong to $\mathcal{S}$).

So V is reducible and there exist $V_1 \subsetneq V$, $V_2 \subsetneq V$ such that $V = V_1 \cup V_2$.

Since V is a minimal element, we have $V_1, V_2 \notin \mathcal{S}$, i.e.

$$V_1 = \bigcup_{i=1}^{r} V_{1i} \quad \text{and} \quad V_2 = \bigcup_{j=1}^{s} V_{2j}, \quad V_{ij} \text{ irreducible.}$$

Therefore

$$V = V_1 \cup V_2 = \bigcup_{i=1}^{r} \bigcup_{j=1}^{s} V_{ij}.$$

We get then that any algebraic set may be written as

$$V = V_1 \cup \dots \cup V_m, \quad V_i \text{ irreducible.}$$

If $V_i \subseteq V_j$, for $i \neq j$ then we can throw away V_i.

UNIQUENESS

If V has two decompositions into irreducible components:

$$V = V_1 \cup \dots \cup V_m = W_1 \cup \dots \cup W_n$$

$$\Downarrow$$

$$V \cap V_i = (W_1 \cup \dots \cup W_n) \cap V_i$$

$$\Downarrow$$

$$V_i = (W_1 \cap V_i) \cup \dots \cup (W_n \cap V_i)$$

Then, since V_i is irreducible, $\exists\, j$ such that $W_j \cap V_i = V_i \Rightarrow$
$\Rightarrow V_i \subseteq W_j$.

Similarly, $\exists\, k$ such that $W_j \subseteq V_k$. Hence:

$$V_i \subseteq W_j \subseteq V_k .$$

Then $i = k$ and $V_i = W_j$.