DERIVATIVES OF LOGARITHMIC AND EXPONENTIAL FUCTION (Sec. 3.3)

We will now compute the derivative of a logarithmic function. For this we will use:

- the definition of derivative,
- the laws of logarithm
- the definition of the number e: $e := \lim_{x \to 0} (1+x)^{\frac{1}{x}}$

We will then use the derivative of the logarithm for computing the derivative of the exponential function.

Let $f(x) = \log_a(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\log_a(x+h) - \log_a(x)}{h} =$$

$$= \lim_{h \to 0} \frac{\log_a\left(\frac{x+h}{x}\right)}{h} = \lim_{h \to 0} \frac{1}{h} \cdot \log_a\left(1 + \frac{h}{x}\right) =$$

$\log_a(x) - \log_a(y) = \log_a\left(\frac{x}{y}\right)$

$$= \lim_{h \to 0} \log_a\left(1 + \frac{h}{x}\right)^{\frac{1}{h}} = \lim_{h \to 0} \log_a(1+t)^{\frac{1}{xt}} =$$

$r \log(x) = \log(x^r)$

$\frac{h}{x} = t \quad \left(\frac{1}{h} = \frac{1}{xt}\right)$

$h \to 0 \Rightarrow t \to 0$

$$= \lim_{t \to 0} \log_a\left(\left(1+t\right)^{\frac{1}{t}}\right)^{\frac{1}{x}} = \lim_{t \to 0} \frac{1}{x} \log_a\left(\left(1+t\right)^{\frac{1}{t}}\right) =$$

$$= \frac{1}{x} \log_a\left(\lim_{t \to 0} (1+t)^{\frac{1}{t}}\right) = \frac{1}{x} \log_a e$$

$\log_a$ continuous function

$e = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$

Hence we have:

$f(x) = \log_a x$ is differentiable and $f'(x) = \frac{1}{x} \log_a e$

In particular when $a = e$ we get $(\ln(x))' = \frac{1}{x} \log_e e = \frac{1}{x}$

$$\boxed{\begin{aligned} (\log_a x)' &= \frac{1}{x} \log_a e \\ (\ln x)' &= \frac{1}{x} \end{aligned}}$$

→ Here it appears clear why e is the "most convenient" base for a logarithmic function chain rule

$$\left[\log_a(f(x))\right]' = \frac{1}{f(x)} \log_a e \cdot f'(x) = \frac{f'(x)}{f(x)} \log_a e$$

$$\left[\ln(f(x))\right]' = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

EXERCISE: Find the derivative of the following functions:

1) $\ln(x^2+4x-1)$
2) $\sqrt{\ln(2x)}$
3) $\log(\sin(x^2))$
4) $\left(\ln\left(e^{\cos(2x)}\right)\right)^3$

Solution

1) $$\left[\ln(x^2+4x-1)\right]' \underset{\uparrow \text{ chain rule}}{=} \frac{(x^2+4x-1)'}{x^2+4x-1} = \frac{2x+4}{x^2+4x-1}$$

2) $$\left[\sqrt{\ln(2x)}\right]' = \left[(\ln(2x))^{\frac{1}{2}}\right]' = \frac{1}{2}(\ln(2x))^{-\frac{1}{2}} \cdot (\ln(2x))' =$$

$$= \frac{1}{2}(\ln(2x))^{-\frac{1}{2}} \cdot \frac{(2x)'}{2x} = \frac{1}{\cancel{2}}(\ln(2x))^{-\frac{1}{2}} \cdot \frac{\cancel{2}}{2x} =$$

$$= \frac{1}{2x\sqrt{\ln(2x)}}$$

3) $$(\underset{\uparrow \log_{10}}{\log}(\sin(x^2)))' = \frac{1}{\sin(x^2)} \cdot \log e \cdot (\sin(x^2))' =$$

$$= \frac{1}{\sin(x^2)} \cdot \log e \cdot \cos(x^2) \cdot (x^2)' =$$

$$= \frac{\log e \cdot \cos(x^2) \cdot 2x}{\sin(x^2)}.$$

4) $$\left[\left(\ln\left(e^{\cos(2x)}\right)\right)^3\right]' \underset{\uparrow \ln(e^x)=x}{=} \left[(\cos(2x))^3\right]' = 3(\cos(2x))^2 \cdot (\cos(2x))' =$$

$$= 3(\cos(2x))^2 \cdot (-\sin(2x)) \cdot (2x)' = 3(\cos(2x))^2 \cdot (-\sin(2x)) \cdot 2 =$$

$$= -6(\cos(2x))^2 \cdot \sin(2x).$$

We will now compute the derivative of the exponential function by using a method called "logarithmic differentiation".

Logarithmic differentiation and derivative of the exponential function

Let $f(x) = a^x$. We want to compute the derivative of f.

For this we will use a method called logarithmic differentiation that we will present in several steps:

① Set $y = f(x)$

$$y = a^x$$

② Take the natural logarithm both sides in the equation $y = f(x)$ and use the laws of logarithm to simplify your right hand expression.

$$\ln(y) = \ln(a^x)$$

$\ln(x^r) = r\ln(x)$

$$\ln(y) = x\ln(a)$$

③ Differentiate both sides implicitly with respect to x

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}\, x\,\underbrace{\ln(a)}_{\text{constant}}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \ln(a)$$

④ Solve your resulting equation for $\frac{dy}{dx}$ and, at the end, do not forget that $y = f(x)$...

$$\frac{dy}{dx} = y\ln(a) \underset{y=a^x}{=} a^x\ln(a)$$

Hence we have:

$f(x) = a^x$ is differentiable and $f'(x) = a^x\ln(a)$

In particular when $a = e$ we get $(e^x)' = e^x\ln(e) \underset{\ln(e)=1}{=} e^x$

$$\boxed{\begin{aligned}(a^x)' &= a^x\ln(a)\\ (e^x)' &= e^x\end{aligned}}$$

→ again we understand here why e is the "most convenient" base for our exponential function.

chain rule
$$\begin{cases} \left[a^{f(x)}\right]' = a^{f(x)} \ln(a) \cdot f'(x) \\ \left[e^{f(x)}\right]' = e^{f(x)} \cdot f'(x) \end{cases}$$

EXERCISE : Find the derivative of the following functions

1) $e^{3\sin(x)}$ 2) $\cos(2^x)$

2) $\dfrac{1}{e^{x^2}}$ 3) $e^{\ln(\sqrt{x})}$

Solution

1) $\left[e^{3\sin(x)}\right]' = e^{3\sin(x)} \cdot (3\sin(x))' =$

$= e^{3\sin(x)} \cdot 3\cos(x)$.

2) $\left[\dfrac{1}{e^{x^2}}\right]' = \left(e^{-x^2}\right)' = e^{-x^2} \cdot (-x^2)' = e^{-x^2}(-2x) =$

$= -2x\,e^{-x^2}$.

3) $\left[\cos(2^x)\right]' = -\sin(2^x) \cdot (2^x)' = \sin(2^x) \cdot 2^x \cdot \ln 2$.

4) $\left[e^{\ln(\sqrt{x})}\right]' = \left[\sqrt{x}\right]' = \dfrac{1}{2\sqrt{x}}$

$e^{\ln(x)} = x$

EXERCISE : Find the derivative of the function:

$$f(x) = x^x$$

Solution

We will solve this exercise in two different ways:

1) LOGARITHMIC DIFFERENTIATION

① $y = f(x)$

$y = x^x$

② TAKE NATURAL LOGARITHM BOTH SIDES + LAWS

$$\ln(y) = \ln(x^x)$$

$$\Downarrow$$

$$\ln(y) = x\ln(x)$$

$\ln(x^r) = r\ln(x)$

③ DIFFERENTIATE WITH RESPECT x

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}\left(x\ln(x)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x)\cdot\ln(x) + x\frac{d}{dx}\ln(x)$$

$$\frac{1}{y}\frac{dy}{dx} = 1\cdot\ln(x) + x\cdot\frac{1}{x}$$

$$\frac{1}{y}\frac{dy}{dx} = \ln(x)+1$$

④ SOLVE FOR $\frac{dy}{dx}$ AND $y = f(x)$...

$$\frac{dy}{dx} = y\left(\ln(x)+1\right) = x^x\left(\ln(x)+1\right)$$

2) <u>II METHOD</u>

$$f(x) = x^x = e^{\ln(x^x)} = e^{x\ln(x)}$$

$x = e^{\ln(x)}$

$$f'(x) = \left(e^{x\ln(x)}\right)' = e^{x\ln(x)}\cdot\left(x\ln(x)\right)' =$$

$$= e^{x\ln(x)}\cdot\left((x)'\ln(x) + x\left(\ln(x)'\right)\right) =$$

$$= e^{x\ln(x)}\cdot\left(1\cdot\ln(x) + x\cdot\frac{1}{x}\right) = x^x\cdot\left(\ln(x)+1\right)$$

Logarithmic differentiation can be used for finding derivatives of complicated functions involving products, quotients, powers. Indeed the law of logarithm allows to simplify a lot the expression:

ex: Find the derivative of the following function:

$$f(x) = \frac{x^6\left(x^3+2x-1\right)^{\sqrt{2}}}{\left(x^2+5\right)^6}.$$

① $y = f(x)$

$$y = \frac{x^6 (x^3 + 2x - 1)^{\sqrt{2}}}{(x^2+5)^6}$$

② TAKE ln BOTH SIDES + APPLY LAWS

$$\ln(y) = \ln\left(\frac{x^6 (x^3 + 2x - 1)^{\sqrt{2}}}{(x^2+5)^6}\right)$$

$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$

$$\ln(y) = \ln\left(x^6 (x^3 + 2x - 1)^{\sqrt{2}}\right) - \ln(x^2+5)^6$$

$\ln(xy) = \ln(x) + \ln(y)$

$$\ln(y) = \ln(x^6) + \ln(x^3+2x-1)^{\sqrt{2}} - \ln(x^2+5)^6$$

$\ln(x^r) = r\ln(x)$

$$\ln(y) = 6\ln(x) + \sqrt{2}\ln(x^3+2x-1) - 6\ln(x^2+5)$$

③ DIFFERENTIATE WITH RESPECT TO x

$$\frac{d}{dx}\left(\ln(y)\right) = \frac{d}{dx}\left(6\ln(x) + \sqrt{2}\ln(x^3+2x-1) - 6\ln(x^2+5)\right)$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{d}{dx}6\ln(x) + \frac{d}{dx}\left(\sqrt{2}\ln(x^3+2x-1)\right) - \frac{d}{dx}\left(6\ln(x^2+5)\right)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{6}{x} + \frac{\sqrt{2}(3x^2+2)}{x^3+2x-1} - \frac{6(2x)}{x^2+5}$$

④ SOLVE FOR $\frac{dy}{dx}$ AND $y = f(x)$

$$\frac{dy}{dx} = y\left(\frac{6}{x} + \frac{\sqrt{2}(3x^2+2)}{x^3+2x-1} - \frac{6(2x)}{x^2+5}\right) =$$

$$= \frac{x^6(x^3+2x-1)^{\sqrt{2}}}{(x^2+5)^6}\cdot\left(\frac{6}{x} + \frac{\sqrt{2}(3x^2+2)}{x^3+2x-1} - \frac{6(2x)}{x^2+5}\right) =$$