DERIVATIVES AND SHAPES OF GRAPHS (Sec. 4.3)

As we saw already in the previous class, the Mean Value Theorem enables us to deduce information about a function from information about its derivative.

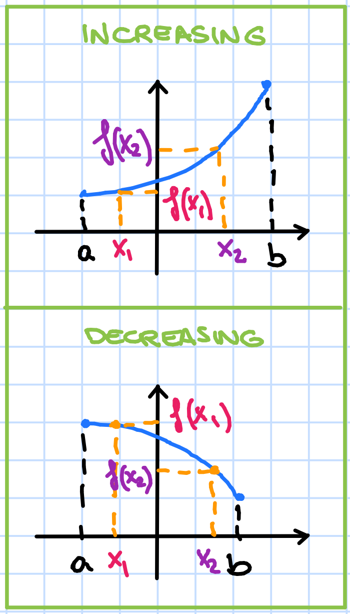
It is not surprising that f' contains information about f: indeed, since $f'(x)$ is the slope of the tangent line to the curve $y=f(x)$ at each point $(x, f(x))$, it tells us the direction in which the curve proceeds at each point.

In the same way, also f'' contains information on f' and consequently on f.

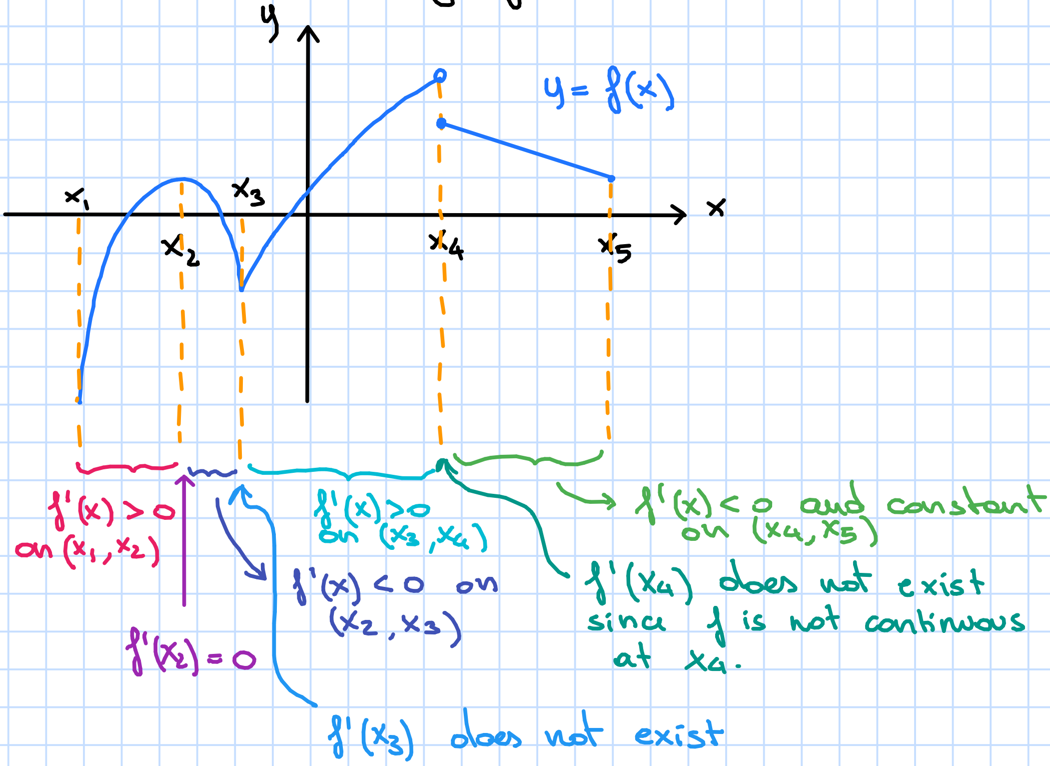
What does f' say about f?

We recall first the following definition.

Def:
- A function f is said to be strictly increasing on an interval (a,b) if for all $x_1, x_2 \in (a,b)$ such that $x_1 < x_2$ then $f(x_1) < f(x_2)$.
- A function f is said to be strictly decreasing on an interval (a,b) if for all $x_1, x_2 \in (a,b)$ such that $x_1 < x_2$ then $f(x_1) > f(x_2)$.



Let us consider the following graph of a function f:



On the previous graph we remark that:

- when $f'(x) > 0$, the function is strictly increasing
- when $f'(x) < 0$, the function is strictly decreasing
- when $f'(x)$ changes sign (from positive to negative or from negative to positive) then we have a local maximum or minimum point (unless the function is not continuous).
- when $f'(x)$ is constant, then the graph of f is a line.

We have indeed the following result.

INCREASING DECREASING TEST

Let f be a function which is differentiable on (a,b).

- If $f'(x) > 0$ on (a,b) then $f(x)$ is increasing on (a,b).
- If $f'(x) < 0$ on (a,b) then $f(x)$ is decreasing on (a,b).

$f'(x)$	$+$	$-$
$f(x)$	↗	↘

Proof

We will prove only the first assertion. The second one can be proven similarly.

We assume that $f'(x) > 0$.

Let $x_1, x_2 \in (a,b)$ with $x_2 > x_1$. Since f is differentiable on (a,b) and $(x_1, x_2) \subseteq (a,b)$, then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2).

Then, by the Mean Value Theorem, there exists c in (x_1, x_2) such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Now, by hypothesis $f'(c) > 0$. Thus:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0 \underset{x_2 - x_1 > 0}{\Longleftrightarrow} f(x_2) - f(x_1) > 0 \Longleftrightarrow f(x_2) > f(x_1).$$

Since x_1 and x_2 are any numbers in (a,b) with $x_2 > x_1$, by definition we obtain that f is strictly increasing on (a,b).

$\square$

A consequence of the previous result is the first derivative test:

FIRST DERIVATIVE TEST

Suppose that c is a critical number of a continuous function.

- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

		c
$f'(x)$	$+$	$-$
$f(x)$	↗	↘

↓ f has a local MAXIMUM at c

		c
$f'(x)$	$-$	$+$
$f(x)$	↘	↗

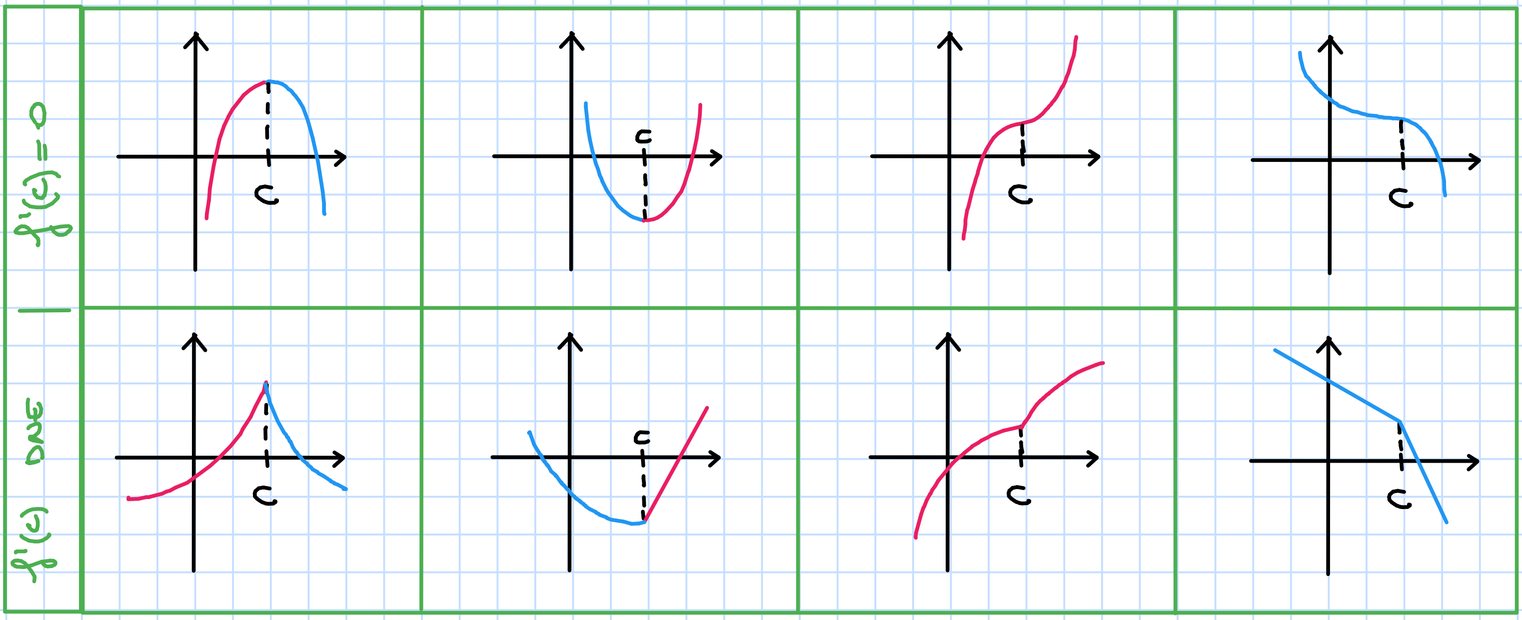
↓ f has a local MINIMUM at c

		c
$f'(x)$	$+$	$+$
$f(x)$	↗	↗

↓ NO MAX/MIN

		c
$f'(x)$	$-$	$-$
$f(x)$	↘	↘

↓ NO MAX/MIN.



→ Note that for the function whose graph is in the first page only $f(x_2)$ and $f(x_3)$ are local max/min. for f. The value $f(x_4)$ is neither local max. nor min. Indeed the first derivative test cannot be applied since f is not continuous at x_4.

EXERCISE

Consider the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5.$$

(1) Find the critical points of f.

(2) Find the intervals of increase / decrease of f.

(3) Find the local maximum and minimum values of f.

Solution

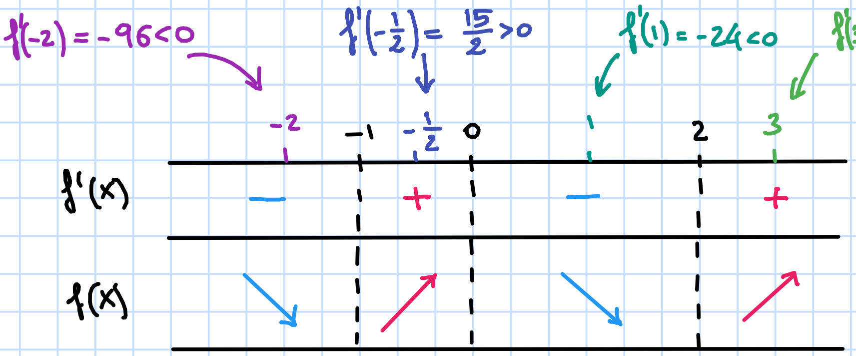
(1) Since f is a polynomial then it is differentiable everywhere. Consequently c is a critical point of f if and only if $f'(c) = 0$.

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1)$$

Now $f'(x) = 0 \iff 12x(x-2)(x+1) = 0 \iff x = 0,\ x = 2$ or $x = -1$.

(2) The critical numbers -1, 0 and 2 divide the real line into 4 intervals: $(-\infty, -1)$, $(-1, 0)$, $(0, 2)$, $(2, \infty)$.

We determine in each one of these intervals the sign of $f'(x)$.



Since the derivative is continuous, its sign over an interval is given by the sign of whatever value in that interval.

Then f is increasing over $(-\infty, -1) \cup (0,2)$ and increasing over $(-1,0) \cup (2, \infty)$.

(3) From (2) we get also that f has local minimum values at -1 and 2 and a local maximum value at 0:

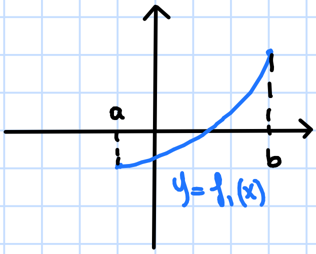
$f(-1) = 0$ LOC. MIN. VALUE

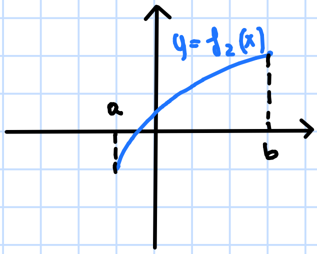
$f(0) = 5$ LOC. MAX. VALUE

$f(2) = -27$ LOC. MIN. VALUE

What does f'' say about f ?

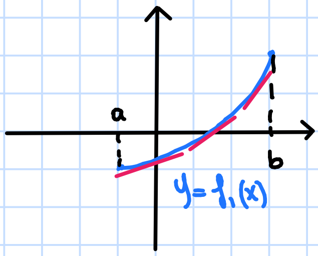
Consider the graphs of the following functions f_1 and f_2.

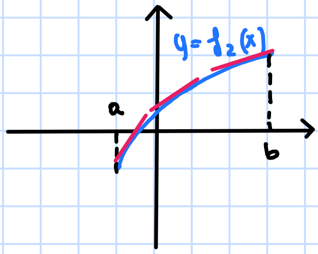




Both f_1 and f_2 are increasing on (a,b) but they look different because they bend in different directions.

We notice that for the function f_1 the tangent line at each point lies under the curve, while for the function f_2 it lies above the curve.





We say that f_1 is "concave upward" on (a,b) while f_2 is "concave downward" on (a,b).

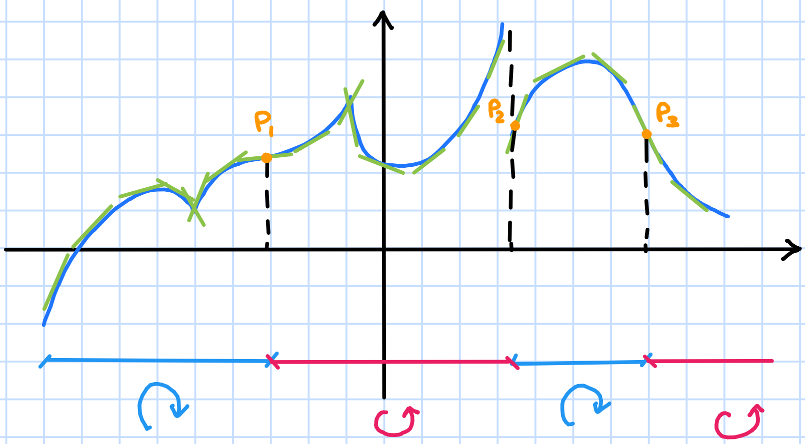
We have the following definition:

Def: A function f is said to be concave upward on (a,b) if the graph of f lies above all its tangents on (a,b).

A function f is said to be concave downward on (a,b) if the graph of f lies below all its tangents on (a,b).

CONCAVE UPWARD

CONCAVE DOWNWARD



Def: A point $P(x_0, f(x_0))$ on the graph of a function f is called an *inflection point* if f is continuous at x_0 and the graph changes from concave upward to concave downward or from concave downward to concave upward at P.

Remark: ① In the previous graph only P_1 and P_3 are inflection points since, even if the function changes from concave upward to concave downward at P_2, it is not continuous at P_2

② If the graph of a function has a tangent at an inflection point $(x_0, f(x_0))$, i.e. $f'(x_0)$ exists, then the graph crosses its tangent there

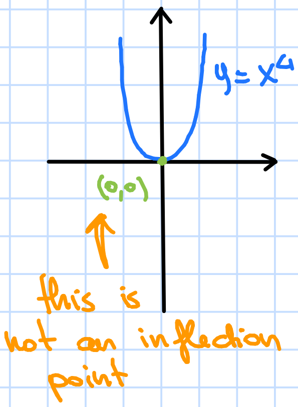
For determining the "concavity" of a function we use the second derivative!

CONCAVITY TEST

- If $f''(x) > 0$ for all x in an interval (a,b), then the graph of f is concave upward on (a,b)
- If $f''(x) < 0$ for all x in an interval (a,b), then the graph of f is concave downward on (a,b)

Remarks:
- If $f''(x)$ exists for all x in (a,b) and $(x_0, f(x_0))$ is an inflection point, with $x_0 \in (a,b)$, then $f''(x_0) = 0$.

  Warning: the converse is not true. Indeed for $f(x) = x^4$ we have:



$f'(x) = 4x^3$

$f''(x) = 12x^2$

So $f''(0) = 0$ but $(0,0)$ is not an inflection point because the graph of $f(x)$ is concave upward on $(-\infty, \infty)$ ($f''(x) \geq 0$ for all x).

- Note that if $f''(x) > 0$ (resp. $f''(x) < 0$) then $f'(x)$ increases (resp. decreases) on (a,b), i.e. the slopes of the tangent lines increase (resp. decrease).

EXERCISE

Consider the function $f(x) = x^4 - x^3$.

(1) Find the interval(s) on which $f(x)$ is concave up / down.

(2) Find the coordinates of the inflection points of f.

Solution

(1) We have:

$$f'(x) = 4x^3 - 3x^2$$

$$f''(x) = 12x^2 - 6x = 6x(2x-1)$$

We have to study the sign of $f''(x)$. First of all let us find the zeros of f'':

$$f''(x) = 0 \iff 6x(2x-1) = 0 \iff x = 0 \text{ or } x = \frac{1}{2}.$$

So:

$f''(-1) = 18 > 0$ $\quad$ $f''\left(\frac{1}{4}\right) = -\frac{6}{8} < 0$ $\quad$ $f''(1) = 6 > 0$

x	-1	0	$\frac{1}{4}$	$\frac{1}{2}$	1
$f''(x)$	$+$		$-$		$+$
$f(x)$	$\cup$		$\cap$		$\cup$

In conclusion f is concave up on $(-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$ and concave down on $\left(0, \frac{1}{2}\right)$.

(2) • At 0 the function is continuous and changes from concave up to concave down.
So $(0, f(0)) = (0,0)$ is an inflection point.

• At $\frac{1}{2}$ the function is continuous and changes from concave down to concave up.

So $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, -\frac{1}{16}\right)$ is an inflection point.

There exists a second method for finding local max/min values which uses the second derivative of the function and is based on the following result:

SECOND DERIVATIVE TEST

Suppose f'' is continuous near c.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Proof

We will prove only the first assertion.

We assume that $f''(c) > 0$. Since $f''(x)$ is continuous near c, then there exists an interval I with c in I such that $f''(x) > 0$ for all x in I.

This implies that $f'(x)$ is increasing on I. Since $f'(c) = 0$ we have that $f'(c) < 0$ before c and $f'(c) > 0$ after c. By the first derivative test f has a local minimum at c.

$\square$

Example: In a previous exercise we showed by the first derivative test that the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

has local minimum values at -1 and 2 and a local maximum value at 0.

We can also prove it with the second derivative test.

We have:

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$$

$$f''(x) = 36x^2 - 24x - 24$$

So:

- $f'(-1) = 0$ and $f''(-1) = 36 + 24 - 24 = 36 > 0 \to f(-1)$ loc. min.
- $f'(0) = 0$ and $f''(0) = -24 < 0 \Rightarrow f(0)$ loc. max.
- $f'(2) = 0$ and $f''(2) = 36 \cdot 4 - 24 \cdot 2 - 24 > 0 \Rightarrow f(2)$ loc. min,

Remark: Note that the second derivative test does not always work. Indeed in the case where $f'(c)=0$ and $f''(c)=0$ we can not conclude anything.

Moreover sometimes the computation of the second derivative can be quite long.

This is why most of the times it preferable to use the first derivative test.

<u>example</u>: • $f(x) = x^3$

$f'(x) = 3x^2$, $f''(x) = 6x$

We have $f'(0)=0$ and $f''(0)=0$.
The first derivative test tells us that f has neither a local max nor min at 0 ($f'(x)$ is positive before and after 0).

• $f(x) = x^4$

$f'(x) = 4x^3$, $f''(x) = 12x^2$

We have $f'(0)=0$ and $f''(0)=0$, but this time f has a local minimum at 0 (f' is negative before 0 and positive after 0).