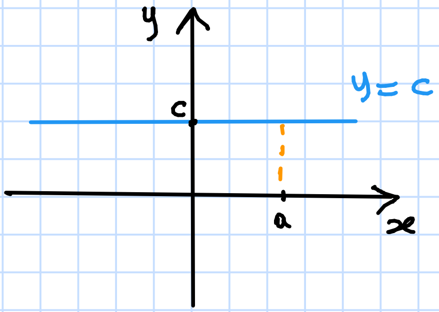
Calculating limits (Sec. 1.4 of the book)

So far we have seen how to compute limits intuitively (table of values) or visually (from a graph).
In this section we will study how to compute the limit of a function algebraically, that is, by using uniquely its algebraic expression.

Let a be a real number.
Let us consider first the following two easy cases:

- $f(x) = c$, where c is a constant (= a real number)

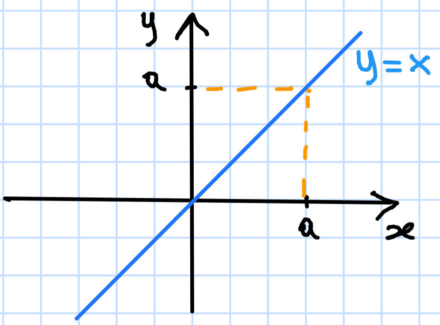


From the graph (and also algebraically) it is clear that while x approaches a the function $f(x) = c$ gets closer and closer to c (actually it is equal to c for all real numbers).

Then we have

$$\boxed{\lim_{x \to a} c = c}$$ SPECIAL LIMIT 1

- $f(x) = x$



Again, when x approaches a the function $f(x) = x$ gets either side closer and closer to a, that is

$$\boxed{\lim_{x \to a} x = a}$$ SPECIAL LIMIT 2

Moreover limits satisfy the following properties:
(in order to prove them you need the more formal definition with ε and δ)

LIMIT LAWS

Let a be a real number and f and g two functions defined near a (except possibly at a). Suppose also that

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

exist (this means that for both functions left-hand and right-hand limit when x approaches a are equal and given by a finite real number - i.e. not ∞).

Then:

1. $\lim_{x\to a}\left[f(x)+g(x)\right]=\left[\lim_{x\to a} f(x)\right]+\left[\lim_{x\to a} g(x)\right]$
2. $\lim_{x\to a}\left[f(x)-g(x)\right]=\left[\lim_{x\to a} f(x)\right]-\left[\lim_{x\to a} g(x)\right]$
3. $\lim_{x\to a}\left[c f(x)\right] = c\left[\lim_{x\to a} f(x)\right]$
4. $\lim_{x\to a}\left[f(x)\, g(x)\right] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right]^{*}$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

* As a consequence we have also:

6. $\lim_{x\to a}\left[f(x)\right]^n = \lim_{x\to a}\left[\underbrace{f(x)\cdots f(x)}_{n \text{ times}}\right] = \underbrace{\left[\lim_{x\to a} f(x)\right]\cdots\left[\lim_{x\to a} f(x)\right]}_{n \text{ times}} = \left[\lim_{x\to a} f(x)\right]^n$

Remark: When for example we write:

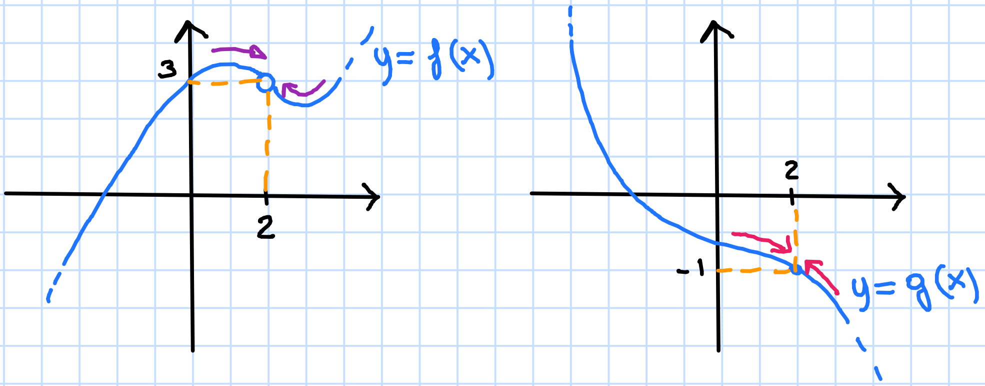
$$\lim_{x\to a}\left[f(x)+g(x)\right]=\left[\lim_{x\to a} f(x)\right]+\left[\lim_{x\to a} g(x)\right]$$

we are saying that:

"the limit of the sum is equal to the sum of the limits".

So, if we are able to compute the limits separately for f and for g then we can also compute the limit of the sum $f+g$.

Let us consider the following example:



Here we have

$$\left.\begin{array}{l}\lim\limits_{x\to 2} f(x) = 3 \\ \lim\limits_{x\to 2} g(x) = -1\end{array}\right\} \Rightarrow \lim_{x\to 2}\left[f(x)+g(x)\right] = \left[\lim_{x\to 2} f(x)\right] + \left[\lim_{x\to 2} g(x)\right] =$$

$$= 3 + (-1) = 2$$

By applying the limit laws to the special limits 1 and 2 we can compute limits of more complicated functions:

ex. 1 : polynomial

$$\lim_{x\to 0}(x^3 - 2x + 3) \underset{①+②}{=} \lim_{x\to 0}(x^3) - \lim_{x\to 0}(2x) + \lim_{x\to 0} 3 =$$

Notation: when in computing an expression we have to go to a new line we put "=" at the end of the line and at the beginning of the new one

$$\underset{⑥+③}{=} \left(\lim_{x\to 0} x\right)^3 - 2\lim_{x\to 0}(x) + \lim_{x\to 0} 3 =$$

$$= (0)^3 - 2\cdot 0 + 3 = 3$$

special limit 1 + special limit 2

We remark that we would have got the same result if we had just plugged in...
We will see that this is not a coincidence!

ex. 2 : rational function

$$\lim_{x \to -1} \frac{x^2-1}{x} \underset{(5)}{=} \frac{\lim_{x\to -1}(x^2-1)}{\lim_{x\to -1}(x)} \underset{(1)+(2)}{=} \frac{\lim_{x\to -1} x^2 - \lim_{x\to -1} 1}{\lim_{x\to -1} x} =$$

$$\underset{(6)}{=} \frac{\left(\lim_{x\to -1} x\right)^2 - \lim_{x\to -1} 1}{\lim_{x\to -1} x} \underset{\text{special limit 1 + special limit 2}}{=} \frac{(-1)^2-1}{-1} = \frac{0}{-1} = 0$$

Again, if we had just plugged -1 into $\frac{x^2-1}{x}$ we would have found the same result.

Indeed we have: if $P(x)$ and $Q(x)$ are polynomials and $Q(a) \neq 0$ (with a a real number) then:

$$\lim_{x\to a} P(x) \underset{\text{plug-in}}{=} P(a) \quad \text{and} \quad \lim_{x\to a} \frac{P(x)}{Q(x)} \underset{\text{plug-in}}{=} \frac{P(a)}{Q(a)}$$

It is possible to show that also the trigonometric functions ($\sin(x)$, $\cos(x)$, $\tan(x)$, $\cot(x)$) enjoy the "plug-in property" (if a is in their domain)

We can summarize this in the following statement

DIRECT SUBSTITUTION PROPERTY (or "PLUG-IN PROPERTY")

If f is a polynomial, or a rational function, or a trigonometric function (or a "combination of those") and a is in the domain of f, then:

$$\lim_{x\to a} f(x) \underset{\text{plug-in}}{=} f(a)$$

This implies that each time that we have to compute a limit <u>we have first to plug-in</u>:

- if we get a number $L \Rightarrow L$ is the value of the limit
- if "the plug-in" does not return a value (for example we get $\frac{0}{0}, \frac{1}{0}, \ldots$) then we have to do <u>more work</u>!

Recall that $\frac{0}{1}$ (and more in general $\frac{0}{\text{a number} \neq 0}$) equals 0!!

ex:

When we plug-in we can remove "$\lim_{x\to 1}$"

- $$\lim_{x\to 1} \frac{\sin\left(\frac{\pi}{2}x\right)+1}{2x^2+3x-1} = \frac{\sin\left(\frac{\pi}{2}\cdot 1\right)+1}{2(1)^2+3\cdot 1-1} = \frac{1+1}{2+3-1} = \frac{2}{4} = \frac{1}{2}$$

plug-in

since we get a number, this is the value of the limit.

Notation: we put here quotation marks since we are not allowed to divide by 0

- $$\lim_{x\to 2} \frac{x^2-4}{x^2+x-6} = \frac{(2)^2-4}{(2)^2+2-6} = \text{"}\frac{0}{0}\text{"}$$: we need more work!

plug-in

This "more work" that we need will be different depending on the kind of function we are dealing with.

We will try to provide some classical examples, but the world of limits is sooo large to be reduced to few cases. Sometimes you really have to be creative!

In each of the following cases we want to compute $\lim_{x\to a} f(x)$, where a is a number (strategies will be different for ∞)

① f is a rational function and by plugging-in a we get "$\frac{0}{0}$" $\to$ factorize + simplify (then plug-in again)

In this case $f(x) = \frac{P(x)}{Q(x)}$, with P and Q two polynomials.

We know that $P(a)=0$ and $Q(a)=0$ (since $f(a) = \text{"}\frac{0}{0}\text{"}$).

Thanks to algebra we are sure that $(x-a)$ is a factor for both $P(x)$ and $Q(x)$ and so we can simplify.

Let us show this on an example (the previous one)

recall: $a^2-b^2=(a+b)(a-b)$

$$\lim_{x\to 2} \frac{x^2-4}{x^2+1\cdot x-6} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x+3)} = \lim_{x\to 2} \frac{x+2}{x+3} = \frac{2+2}{2+3} = \frac{4}{5}$$

plug-in

$-2\cdot 3=-6$

$-2+3=1$

Do not forget to write this again! If you forget it is a mistake, since you are writing that a limit (which is a number) equals a function (which is in general not constant)

We can simplify since we are studying the behaviour of the function near 2 but not at 2! Then $x-2\neq 0$ and we can divide.

This is the consequence of a more general result:

Prop: If f and g are two functions such that $f(x)=g(x)$ for all $x \neq a$ the

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$$

provided these limits exist.

Indeed in our case we have:

$$\underbrace{\frac{x^2-4}{x^2+x-6}}_{f(x)} = \underbrace{\frac{x+2}{x+3}}_{g(x)} \quad \text{for all } x \neq \underset{a}{2}$$

$$\xRightarrow{\text{prop.}} \lim_{x\to 2} \frac{x^2-4}{x^2+x-6} = \underbrace{\lim_{x\to 2} \frac{x+2}{x+3}}_{\substack{\text{easier to compute} \\ \text{(just plug-in)}}} = \frac{4}{5}$$

② The variable in the function appears under a square root and when we plug in a we get "$\frac{0}{0}$"

$\longrightarrow$ multiply by the conjugate, simplify, plug in again

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x^2+x-2} \underset{\text{plug in}}{=} \frac{1-1}{1+1-2} = "\frac{0}{0}"$$

$(a+b)(a-b) = a^2-b^2$

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x^2+x-2} = \lim_{x\to 1} \frac{\sqrt{x}-1}{(x+2)(x-1)} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \lim_{x\to 1} \frac{\cancel{x-1}}{(x+2)\cancel{(x-1)}(\sqrt{x}+1)} =$$

multiply numerator and denominator by the conjugate of $\sqrt{x}-1$ (since we are near 1 and $\sqrt{x}+1 \neq 0$ near 1, we are fine)

the conjugate is obtained by changing the sign in the middle of the two terms

$$= \lim_{x\to 1} \frac{1}{(x+2)(\sqrt{x}+1)} \underset{\text{plug in}}{=} \frac{1}{(1+2)(\sqrt{1}+1)} = \frac{1}{6}$$

Remark: If the square root is in the denominato the technique is the same

ex: $$\lim_{t\to 1} \frac{t-1}{\sqrt{2t}-\sqrt{2}} \cdot \frac{\sqrt{2t}+\sqrt{2}}{\sqrt{2t}+\sqrt{2}} = \lim_{t\to 1} \frac{(t-1)(\sqrt{2t}+\sqrt{2})}{2t-2} = \lim_{t\to 1} \frac{\cancel{(t-1)}(\sqrt{2t}+\sqrt{2})}{2\cancel{(t-1)}} \underset{\text{plug in}}{=} \frac{2\sqrt{2}}{2} = \sqrt{2}$$

③ Function with sine inside and the argument of sine approaching zero
↓ (in general, not always)
Use the special limit $\boxed{\lim_{x \to 0} \frac{\sin(x)}{x} = 1}$ ← learn this, please!

Warning! Before applying the special limit we have to make sure that both the argument of the sine and the denominator are approaching zero.

ex: $\lim_{x \to 0} \frac{\sin(x-\pi)}{x-\pi} = \frac{\sin(-\pi)}{-\pi} = \frac{0}{-\pi} = 0$

plug in

Indeed we can not apply the special limit (that would return a different, wrong answer) because:

$\underbrace{\text{when } x \to 0 \text{ then } x-\pi \to -\pi}$

In other terms $\lim_{x \to 0} x-\pi = -\pi \neq 0$

But

$\lim_{x \to \pi} \frac{\sin(x-\pi)}{x-\pi} = 1$, because of the special limit

0 when $x \to \pi$

0 when $x \to \pi$

ex: $\lim_{x \to 0} \frac{\sin(7x)}{4x} = \lim_{x \to 0} \frac{\sin(7x)}{4x} \cdot \frac{7}{7} = \lim_{x \to 0} \frac{7}{4} \frac{\sin(7x)}{7x} =$

here we can not apply the special limit directly because the argument of the sine (7x) and the denominator are different. ⇒

we multiply and divide by 7 (recall that you have always multiply and divide by the same quantity, otherwise you change your function)

reorganize your function in order to highlight the special limit (do not forget that the product is commutative!)

$= \frac{7}{4} \lim_{x \to 0} \frac{\sin(7x)}{7x} = \frac{7}{4} \cdot 1 = \frac{7}{4}$

$\lim_{x \to a} c\, f(x) = c \cdot \lim_{x \to a} f(x)$
(LIMIT LAW)

Here we can apply the special limit since when $x \to 0$ then $7x \to 0$

ex. Here is a more complicated example involving the special limit $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

$$\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = \lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} \cdot \frac{\cos\theta + 1}{\cos\theta + 1} =$$

$$= \lim_{\theta\to 0} \frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \lim_{\theta\to 0} \frac{\sin^2\theta}{\theta(\cos\theta+1)} = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \cdot \frac{\sin\theta}{\cos\theta + 1} =$$

$\cos^2\theta = (\cos\theta)^2$ same thing different notation

Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$

reorganize (highlight the special limit)

$$= \underbrace{\left(\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\right)}_{=1} \cdot \left(\lim_{\theta\to 0} \frac{\sin\theta}{\cos\theta+1}\right) = 1 \cdot \frac{\sin(0)}{\cos(0)+1} = \frac{0}{1+1} = 0$$

LIMIT LAW (product)

plug in

④ Limit of a piecewise function at its breaking point(s) (or of a function with absolute value)

a function with an absolute value can indeed be rewritten as a piecewise function

Use the following result (already studied in class 2):

$$\boxed{\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L}$$

ex: $f(x) = \begin{cases} \sin(x) + 2, & \text{when } x \le -\frac{\pi}{2} \\ \cos(4x), & \text{when } x > -\frac{\pi}{2} \end{cases}$

$-\frac{\pi}{2}$ is the "breaking point" for this piecewise function, since at that point the function "changes its expression"

Compute $\lim_{x\to -\frac{\pi}{2}} f(x)$.

When we compute the limit of a piecewise function at one of its breaking points, we have to compute separately the left-hand and right-hand limit and use the previous result to conclude.

left-hand limit: $\lim\limits_{x\to -\frac{\pi}{2}^-} f(x) = \lim\limits_{x\to -\frac{\pi}{2}^-} \sin(x)+2 = \sin\left(-\frac{\pi}{2}\right)+2 = 1$

(here $x < -\frac{\pi}{2}$)

right-hand limit: $\lim\limits_{x\to -\frac{\pi}{2}^+} f(x) = \lim\limits_{x\to -\frac{\pi}{2}^+} \cos(4x) = \cos\left[4\cdot\left(-\frac{\pi}{2}\right)\right] = \cos(-2\pi) = 1$

(here $x > -\frac{\pi}{2}$)

Since $\lim\limits_{x\to -\frac{\pi}{2}^-} f(x) = \lim\limits_{x\to -\frac{\pi}{2}^+} f(x) = 1 \Rightarrow \lim\limits_{x\to -\frac{\pi}{2}} f(x) = 1$

ex: $f(x) = \dfrac{|x|}{x} = \begin{cases} \dfrac{-x}{x}, & \text{when } x<0 \\ \dfrac{x}{x}, & \text{when } x>0 \end{cases} = \begin{cases} -1, & \text{when } x<0 \\ 1, & \text{when } x>0 \end{cases}$

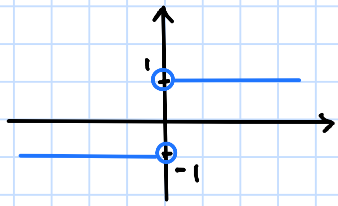
Compute $\lim\limits_{x\to 0} f(x)$:

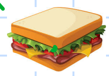
$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} -1 = -1$
$(x<0)$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} 1 = 1$
$(x>0)$

$\Rightarrow$ Since $\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$ then $\lim\limits_{x\to 0} f(x)$ DNE

Indeed the graph of f is



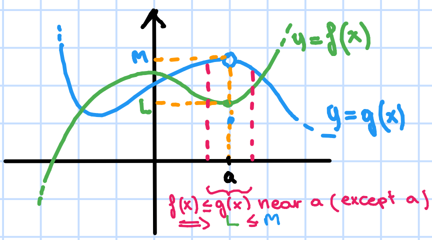
⑤ Squeeze theorem (or " theorem")

Squeeze theorem relies on the following result:

Theorem: If $f(x) \leq g(x)$ near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

→ A visual idea of this theorem:



$\lim\limits_{x\to a} g(x) = M$

$\lim\limits_{x\to a} f(x) = L$

Suppose now that $f(x)$ is a function such that there exist two functions $g(x)$ and $h(x)$ that satisfy:

$$g(x) \le f(x) \le h(x)$$ when x is near a (except possibly at a)

if $\lim_{x\to a} g(x)$, $\lim_{x\to a} f(x)$, $\lim_{x\to a} h(x)$ exist, then the theorem implies that...

$$\lim_{x\to a} g(x) \le \lim_{x\to a} f(x) \le \lim_{x\to a} h(x)$$

If now $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L \Rightarrow L \le \lim_{x\to a} f(x) \le L$,

that is $\lim_{x\to a} f(x) = L$.

Formally we have

SQUEEZE THEOREM (or sandwich theorem)

Let g, f, h be functions defined near a, except possibly at a it'self.
Suppose that for every x near a, except possibly at a we have

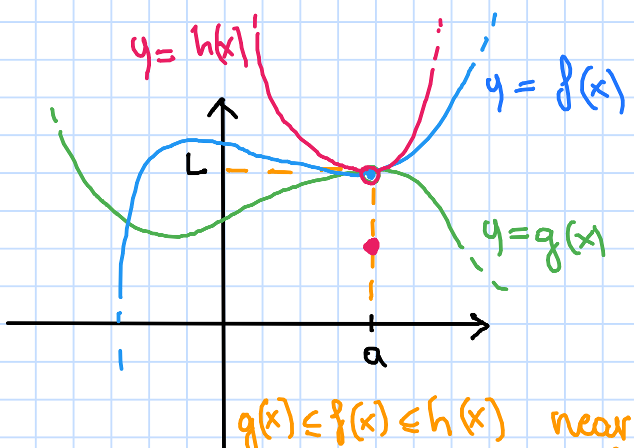
$$g(x) \le f(x) \le h(x)$$

and also suppose that

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L.$$

Then $\lim_{x\to a} f(x) = L$

Visually:



$g(x) \le f(x) \le h(x)$ near a (except a)
and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L \Rightarrow \lim_{x\to a} f(x) = L$

squeeze theorem

Curiosity: In many languages (e.g. French and Italian) the squeeze theorem is also known under the name:

"Two policemen (and a drunk) theorem"

Italian: teorema dei due carabinieri
French: théorème des gendarmes

The analogy is a perfect illustration of squeeze theorem:

If two policemen are escorting a drunk prisoner between them and both officers go to a cell then (regardless the path taken) the prisoner must also end up in the cell.

Squeeze theorem can be useful when computing directly the limit of a function f is hard, but we can actually find two functions g and h that satisfy $g(x) \leq f(x) \leq h(x)$ near a (except possibly at a) and for which the computation is easier and the limit is the same.

This will be clearer with an example:

ex: $\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0$

0 is not in the domain of the function $x^2 \sin\left(\frac{1}{x}\right)$ so the "plug in" will not work.

Now we know that the range of the function $\sin(x)$ is $[-1, 1]$, that is

$$-1 \leq \sin(x) \leq 1 \quad \text{for all } x \in \mathbb{R}$$

and this is true whatever function appears as argument of the sine:

we have just to be careful here

$$-1 \leq \sin(g(x)) \leq 1 \quad \text{for all } x \text{ in the domain of } g$$

So in our case we have:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1 \quad \text{for all } x \neq 0.$$

Thus

$$x^2\cdot(-1) \le x^2\sin\left(\frac{1}{x}\right) \le x^2\cdot 1$$

$$\Updownarrow$$

$$-x^2 \le x^2\sin\left(\frac{1}{x}\right) \le x^2$$

$$\Big\downarrow \text{ when } x\to 0 \qquad \Big\downarrow \text{ when } x\to 0 \qquad \Big\downarrow \text{ when } x\to 0$$

$$0 \le \lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) \le 0$$

squeeze theorem

$$\Longrightarrow \quad \lim_{x\to 0} x^2\sin\left(\frac{1}{x}\right) = 0 .$$

Remark : You can use squeeze theorem for proving the special limit

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

(see page 62 of the book).