CHAIN RULE (Sec. 2.5)

In the previous class we have seen how to differentiate sum, difference, product, qotient of functions.

But do not forget that we can "combine" functions through another operation, called composition.

Indeed at the moment we are not able to find the derivative for functions like:

$$\sin(x^2) \quad \text{or} \quad (\sin(x))^2$$

Recall: If $f(x)$ and $g(x)$ are two functions, then we can define:

$$(f \circ g)(x) = f(g(x))$$
$$(g \circ f)(x) = g(f(x))$$

Note that the operation of composition is in general not comutative:

$$f(g(x)) \neq g(f(x))$$

ex: $f(x) = \sin(x)$, $g(x) = x^2$

$$(f \circ g)(x) = f(g(x)) = f(x^2) = \sin(x^2)$$
$$(g \circ f)(x) = g(f(x)) = g(\sin(x)) = (\sin(x))^2$$

and $\sin(x^2) \neq (\sin(x))^2$

In the process of differentiating a composition of functions the key world is

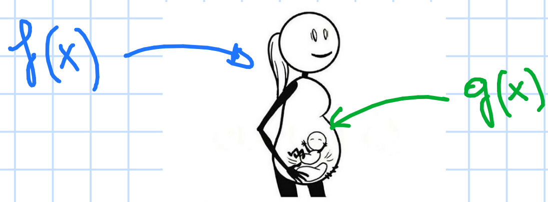
CHAIN RULE ♥

We will see that the only difficulty for applying the chain role is to recognize the outside and the inside function:

$$\underset{\substack{\uparrow \\ \text{OUTSIDE FUNCTION}}}{f}\Big(\underset{\substack{\uparrow \\ \text{INSIDE FUNCTION}}}{g(x)}\Big)$$

"the pregnant woman" "the baby"

You can imagine a composition of functions as a pregnant woman with her baby (cit. Myrto Manolaki)



What is interesting (and a little bit weird) is that with the analogy of "the pregnant woman with a baby", in a composition of functions a baby can also have his/her own baby and so on...

In some sense we can have a composition of functions with several generations:

$$f(g(h(x)))$$

mother — child — grand-child

ex: $f(x) = \sin(x)$, $g(x) = \sqrt{x}$, $h(x) = 3x^2$

$$f(g(h(x))) = f(g(3x^2)) = f(\sqrt{3x^2}) = \sin(\sqrt{3x^2})$$

CHAIN RULE

Let f and g be two differentiable functions. Then:

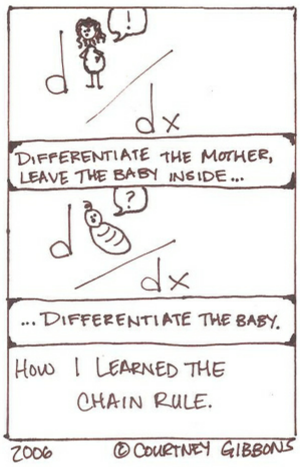
* LAGRANGE NOTATION

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

* LEIBNIZ NOTATION

If $u = g(x)$ and $y = f(u)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$



ex: • $\sin(x^2)$.

OUTSIDE FUNCTION (MOTHER): $f(x) = \sin(x) \Rightarrow f'(x) = \cos(x)$

INSIDE FUNCTION (BABY): $g(x) = x^2 \Rightarrow g'(x) = 2x$

Now "differentiate the mother, leave the baby inside, differentiate the baby." In other words:

differentiate the mother, leave the baby inside, differentiate the baby

$$\left[\sin(x^2)\right]' = \cos(x^2)\, 2x$$

$$f(g(x)) = f'(g(x))\, g'(x)$$

- $(\sin(x))^2$

  OUTSIDE FUNCTION (MOTHER) : $f(x) = x^2 \Rightarrow f'(x) = 2x$

  INSIDE FUNCTION (BABY) : $g(x) = \sin(x) \Rightarrow g'(x) = \cos(x)$

  $$\left[(\sin(x))^2\right]' = 2(\sin(x)) \cdot \cos(x)$$

- $\left[\sqrt{x^2+3}\right]'$

  OUTSIDE FUNCTION (MOTHER) : $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{1}{2\sqrt{x}}$

  INSIDE FUNCTION (BABY) : $g(x) = x^2+3 \Rightarrow g'(x) = 2x$

  $$\left[\sqrt{x^2+3}\right]' = \frac{1}{2\sqrt{x^2+3}} \cdot 2x$$

- Example with Leibniz notation

  We want to differentiate $\tan(3x)$. We set

  $$u = 3x \Rightarrow \frac{du}{dx} = \frac{d}{dx}(3x) = 3$$

  $$y = f(u) = \tan(u) \Rightarrow \frac{dy}{du} = \frac{d}{du}(\tan(u)) = \sec^2 u$$

  $$\frac{d}{dx}\tan(3x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2 u \cdot 3 \underset{u=3x}{=} \sec^2(3x) \cdot 3$$

When do we need chain rule?

a) $3\sin(x)$

b) $\sin(3x)$

c) $\sin(x) \cdot \cos(x)$

d) $\sin(\cos(x))$

e) x^8

f) $(\sqrt{x})^{\frac{1}{4}}$

g) $\frac{1}{\cos(x)}$

Answers

a) No! There is no composition of functions:

$$(3\sin(x))' = 3(\sin(x))' = 3\cos(x)$$

b) YES! We have

$$\sin(3x) = f(g(x)) \quad \text{where} \quad f(x) = \sin(x) \quad \text{and} \quad g(x) = 3x$$

$$\Rightarrow (\sin(3x))' = \cos(3x)\cdot(3x)' = \cos(3x)\cdot 3$$

c) No! This is the product of functions $\sin(x)$ and $\cos(x)$:

$$(\sin(x)\cdot\cos(x))' = (\sin(x))'\cdot\cos(x) + \sin(x)\cdot(\cos(x))' =$$

$$= \cos(x)\cdot\cos(x) + \sin(x)\cdot(-\sin(x)) =$$

$$= \cos^2(x) - \sin^2(x) \underset{\substack{\uparrow \\ \text{trigonometric} \\ \text{formula}}}{=} \cos(2x)$$

d) YES! We have

$$\sin(\cos(x)) = f(g(x)) \quad \text{where} \quad f(x) = \sin(x) \quad \text{and} \quad g(x) = \cos(x)$$

$$\Rightarrow [\sin(\cos(x))]' = \cos(\cos(x))\cdot(\cos(x))' =$$

$$= \cos(\cos(x))\cdot(-\sin(x)) =$$

$$= -\cos(\cos(x))\cdot\sin(x).$$

⚠ Warning: $\cos(\cos(x)) \neq \cos^2(x)$

Indeed if $x = \frac{\pi}{2}$ then:

- $\cos(\cos(\frac{\pi}{2})) = \cos(0) = 1$
- $\cos^2(\frac{\pi}{2}) = (\cos(\frac{\pi}{2}))^2 = 0^2 = 0$

Hence the two functions are different since they do not have the same values at the same numbers.

e) No!

$$(x^8)' = 8x^7$$

f) YES and NO!

We could apply chain rule:

$$\left[\left(\sqrt{x}\right)^{\frac{1}{4}}\right]' = \frac{1}{4}\left(\sqrt{x}\right)^{\frac{1}{4}-1} \cdot \left(\sqrt{x}\right)' = \frac{1}{4}\left(\sqrt{x}\right)^{-\frac{3}{4}} \cdot \frac{1}{2\sqrt{x}} =$$

$$= \frac{1}{4}\left(\left(x\right)^{\frac{1}{2}}\right)^{-\frac{3}{4}} \cdot \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{8} x^{-\frac{3}{8}} \cdot x^{-\frac{1}{2}} =$$

$$= \frac{1}{8} x^{\left(-\frac{3}{8}-\frac{1}{2}\right)} = \frac{1}{8} x^{-\frac{7}{8}} = \frac{1}{8\sqrt[8]{x^7}}$$

But in this case it is easier to first rewrite the function $\left(\sqrt{x}\right)^{\frac{1}{4}}$ as a power function:

$$\left[\left(x\right)^{\frac{1}{2}}\right]^{\frac{1}{4}} = x^{\frac{1}{8}} \Rightarrow \left[x^{\frac{1}{8}}\right]' = \frac{1}{8} x^{\frac{1}{8}-1} = \frac{1}{8} x^{-\frac{7}{8}} = \frac{1}{8\sqrt[8]{x^7}}$$

luckily we get the same result!

g) YES and NO!

We can compute this derivative in two ways:

- we can use the quotient rule:

$$\left(\frac{1}{\cos(x)}\right)' = \frac{(1)' \cdot \cos(x) - 1 \cdot (\cos(x))'}{\cos^2(x)} = \frac{0 \cdot \cos(x) - (-\sin(x))}{\cos^2(x)} =$$

$$= \frac{\sin(x)}{\cos^2(x)}$$

- we can use chain rule if we note that:

$$\frac{1}{\cos(x)} = f(g(x)), \text{ where } f(x) = \frac{1}{x} \text{ and } g(x) = \cos(x).$$

Now $f'(x) = -\frac{1}{x^2}$ and $g'(x) = -\sin(x)$, so that:

$$\left(\frac{1}{\cos(x)}\right)' = -\frac{1}{\cos^2(x)} \cdot (-\sin(x)) = \frac{\sin(x)}{\cos^2(x)}$$

and again, luckily we get the same result!