L'HOSPITAL'S RULE (Sec. 3.7)

In this section we will deal again with the computation of limits. Now we dispose of new tools compared to the beginning of the course, when we used only the definition of limit.

Let us start by reviewing the definition of an "indeterminate form":

Def: A form of limit is said to be indeterminate when knowing the limit behavior of individual parts of the expression is not sufficient to actually determine the overall limit.

There exist 7 different indeterminate forms:

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad 0 \cdot \infty, \quad \infty - \infty, \quad 0^0, \quad 1^\infty, \quad \infty^0$$

Examples

- $$\lim_{x \to 2} \frac{x^2 - 4x + 4}{x^2 - 4} = \text{“}\frac{0}{0}\text{”}$$

  In the beginning of the course we solved this kind of limit by factorizing numerator and denominator:

  $$\lim_{x \to 2} \frac{x^2 - 4x + 4}{x^2 - 4} = \lim_{x \to 2} \frac{(x-2)^{\cancel{2}}}{(x+2)\cancel{(x-2)}} = \lim_{x \to 2} \frac{x-2}{x+2} = \frac{0}{4} = 0$$

- $$\lim_{x \to 0} \frac{\sin(x)}{x} = \text{“}\frac{0}{0}\text{”}$$

  By a geometric argument it is possible to show that:

  $$\lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad \text{(special limit)}$$

- $$\lim_{x \to \infty} \frac{x^2 - 2}{-x^3 - x + 1} = \text{“}\frac{\infty}{\infty}\text{”}$$

  This is the limit at infinity of a rational function. We proved that when the degree of the denominator is greater than the degree of the numerator then the result of the limit is 0:

  $$\lim_{x \to \infty} \frac{x^2 - 2}{-x^3 - x + 1} = 0.$$

For each one of the previous examples there exist "specific techniques" for computing the limit.

But what about the following cases?

$$\lim_{x\to 0} \frac{e^x-1}{x} = \text{"}\frac{0}{0}\text{"}$$

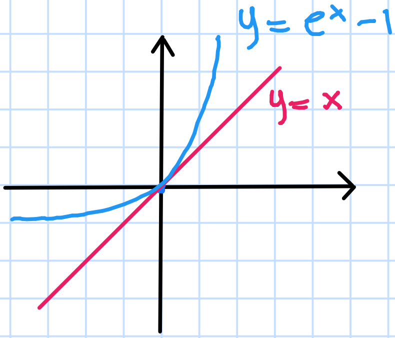
$$\lim_{x\to\infty} \frac{\ln(x)}{x-1} = \text{"}\frac{\infty}{\infty}\text{"}$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \text{"}\frac{\infty}{\infty}\text{"}$$

Let us consider the first limit: $\lim\limits_{x\to 0} \frac{e^x-1}{x}$ and let us set:

$f(x) = e^x - 1$ (numerator)

$g(x) = x$ (denominator)



The linearizations of f and g near 0 are:

$$L_f(x) = f(0) + f'(0)(x-0) = 0 + 1\cdot(x-0) = x$$

$f'(x) = e^x$

$$L_g(x) = g(0) + g'(0)(x-0) = 0 + 1\cdot(x-0) = x$$

Then near 0 we have $f(x) \sim L_f(x)$ and $g(x) \sim L_g(x)$ so that:

$$\lim_{x\to 0} \frac{e^x-1}{x} \sim \lim_{x\to 0} \frac{L_f(x)}{L_g(x)} = \lim_{x\to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1.$$

So 1 is the result that we expect for this limit.

More in general, assume that f and g are two function such that:

- $f(a) = g(a) = 0$
- the derivative functions f' and g' are continuous.
  (in particular f and g are differentiable)
- $g'(x) \neq 0$ near a.

We are interested in computing $\lim_{x\to a} \frac{f(x)}{g(x)}$.

If we linearize near a we have:

$$f(x) \sim L_f(x) = f(a) + f'(a)(x-a) \underset{\substack{\uparrow \\ f(a)=0}}{=} f'(a)(x-a);$$

$$g(x) \sim L_g(x) = g(a) + g'(a)(x-a) \underset{\substack{\uparrow \\ g(a)=0}}{=} g'(a)(x-a).$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} \sim \lim_{x\to a} \frac{L_f(x)}{L_g(x)} = \lim_{x\to a} \frac{f'(a)\cancel{(x-a)}}{g'(a)\cancel{(x-a)}} = \frac{f'(a)}{g'(a)} \underset{\substack{\uparrow \\ f',g' \text{ continuous}}}{=} \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

In conclusion, under particular assumptions we have:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

This result is called "L'Hospital's rule" and is stated more formally and in a more general case as follows:

Theorem (L' HOSPITAL'S RULE)

Suppose f and g are two differentiable functions and $g'(x) \neq 0$ near a (except possibly at a).

Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0 \qquad \left(\lim_{x\to a} \frac{f(x)}{g(x)} = \text{"}\frac{0}{0}\text{"} \right)$$

or

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty \left(\lim_{x\to a} \frac{f(x)}{g(x)} = \text{"}\frac{\infty}{\infty}\text{"} \right).$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or ∞, $-\infty$)

Remark: L'Hospital's rule applies also in the cases $a = \infty$ or $a = -\infty$.

Curiosity: L'Hospital's rule is written sometimes with a different orthography, depending whether we follow the modern or the older French spelling:

L'Hospital's rule : OLD

L'Hôpital's rule : MODERN

Indeed, in the mid 18th century there was a change in French orthography, where some mute s's were dropped and replaced by the circumflex accent.

Examples

- $$\lim_{x\to 2} \frac{x^2-4x+4}{x^2-4} \underset{\uparrow}{=} \lim_{x\to 2} \frac{(x^2-4x+4)'}{(x^2-4)'} = \lim_{x\to 2} \frac{2x-4}{2x} = \frac{0}{4} = 0$$

  $\frac{0}{0} \Rightarrow$ H.R. ⇝ each time you apply l'Hospital's rule you have to highlight which indeterminate form you are dealing with.

- $$\lim_{x\to 0} \frac{\sin(x)}{x} \underset{\uparrow}{=} \lim_{x\to 0} \frac{(\sin(x))'}{(x)'} = \lim_{x\to 0} \frac{\cos(x)}{1} = \cos(0) = 1$$

  $\frac{0}{0} \Rightarrow$ H.R.

- $$\lim_{x\to \infty} \frac{x^2-2}{-x^3-x+1} \underset{\uparrow}{=} \lim_{x\to \infty} \frac{2x}{-3x^2-1} \underset{\uparrow}{=} \lim_{x\to \infty} \frac{2}{-6x} = \text{"}\frac{2}{-\infty}\text{"} = 0$$

  $\frac{\infty}{\infty} \Rightarrow$ H.R. $\qquad$ $\frac{\infty}{\infty} \Rightarrow$ H.R.

- $$\lim_{x\to 0} \frac{e^x-1}{x} \underset{\uparrow}{=} \lim_{x\to 0} \frac{(e^x-1)'}{(x)'} = \lim_{x\to 0} \frac{e^x}{1} = \frac{e^0}{1} = 1$$

  $\frac{0}{0} \Rightarrow$ H.R.

- $$\lim_{x\to \infty} \frac{\ln(x)}{x-1} \underset{\uparrow}{=} \lim_{x\to \infty} \frac{(\ln(x))'}{(x-1)'} = \lim_{x\to \infty} \frac{\frac{1}{x}}{1} = \lim_{x\to \infty} \frac{1}{x} = 0$$

  $\frac{\infty}{\infty} \Rightarrow$ H.R.

- $$\lim_{x\to \infty} \frac{e^x}{x^2} \underset{\uparrow}{=} \lim_{x\to \infty} \frac{(e^x)'}{(x^2)'} = \lim_{x\to \infty} \frac{e^x}{2x} \underset{\uparrow}{=} \lim_{x\to \infty} \frac{e^x}{2} = \infty$$

  $\frac{\infty}{\infty} \Rightarrow$ H.R. $\qquad$ $\frac{\infty}{\infty} \Rightarrow$ H.R.

Remark that l'Hospital's rule can be only applied when we are dealing with the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

Nevertheless, it is also possible to use l'Hospital's rule with the indeterminate forms $0\cdot\infty$, 0^0, 1^∞ and ∞^0. Indeed, in each one of these cases it is possible to reduce the indeterminate form to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by doing some "manipulations" on the function.

$\boxed{0\cdot\infty}$

Suppose that we want to compute

$$\lim_{x\to a} f(x)\cdot g(x) \qquad (*)$$

where $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=\infty$.

We can rewrite $(*)$ in the following ways:

- $\lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} \frac{f(x)}{\frac{1}{g(x)}} \rightsquigarrow \frac{0}{0}$ ($\frac{0}{\frac{1}{\infty}}$)
- $\lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} \frac{g(x)}{\frac{1}{f(x)}} \rightsquigarrow \frac{\infty}{\infty}$ ($\frac{\infty}{\frac{1}{0}}$)

and we can now apply l'Hospital's rule

Note that both rewriting are correct, but normally one is easier than the other for the computation of the limit.

Example

- $\lim_{x\to 0^+} x^2 \ln(x) \underset{0\cdot\infty}{=} \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x^2}} \underset{\frac{\infty}{\infty}\Rightarrow H.R.}{=} \lim_{x\to 0^+} \frac{(\ln(x))'}{(x^{-2})'} =$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-2(x^{-3})} = \lim_{x\to 0^+} -\frac{1}{2}\cdot\frac{1}{x}\cdot x^3 = \lim_{x\to 0^+} -\frac{x^2}{2} = 0$$

Note that we could also rewrite in the following way:

$$\lim_{x\to 0^+} x^2\ln(x) = \lim_{x\to 0^+} \frac{x^2}{\frac{1}{\ln(x)}} \rightsquigarrow \frac{0}{0}$$

but in this case it is easy to see that l'Hospital's rule would not simplify the computations.

0^0 ① $\qquad 1^\infty$ ② $\qquad \infty^0$ ③

Suppose that we want to compute $\lim_{x\to a} f(x)^{g(x)}$ where:

① $\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$;

② $\lim_{x\to a} f(x)=1$ and $\lim_{x\to a} g(x)=\infty$;

③ $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=0$.

In each one of these cases we rewrite:

$$\lim_{x\to a} f(x)^{g(x)} \underset{x=e^{\ln(x)}}{=} \lim_{x\to a} e^{\ln\left(f(x)^{g(x)}\right)} \underset{\ln(x^r)=r\ln(x)}{=} \lim_{x\to a} e^{g(x)\cdot\ln(f(x))} =$$

$$\underset{\substack{e^x \text{ is a} \\ \text{continuous function}}}{=} e^{\lim_{x\to a} g(x)\ln(f(x))}$$

Thus, the computation of the initial limit is reduced to the computation of the following one:

$$\lim_{x\to a} g(x)\cdot\ln(f(x)) \quad \begin{aligned} &\xrightarrow{①} 0\cdot\ln(0) = 0\cdot\infty \\ &\xrightarrow{②} \infty\cdot\ln(1) = \infty\cdot 0 \\ &\xrightarrow{③} 0\cdot\ln(\infty) = 0\cdot\infty \end{aligned}$$

We notice that in all three cases the indeterminate form is reduced to $0\cdot\infty$. Hence, we will use the previous technique for computing the limit.

Thus if

$$\lim_{x\to a} g(x)\ln(f(x)) = A \Rightarrow \lim_{x\to a} f(x)^{g(x)} = e^A$$

Examples

- $\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln(x^x)} = \lim_{x\to 0^+} e^{x\ln(x)} = e^{\lim_{x\to 0^+} x\ln(x)}$

Now:

$$\lim_{x\to 0^+} x\ln(x) \underset{0\cdot\infty}{=} \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \underset{\frac{\infty}{\infty} \Rightarrow H.R.}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} =$$

$$= \lim_{x\to 0^+} -\frac{x^2}{x} = \lim_{x\to 0^+} -x = 0.$$

$$\Rightarrow \lim_{x\to 0^+} x^x = e^{\lim_{x\to 0^+} x\ln(x)} = e^0 = 1$$

- $$\lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = \lim_{x\to 0^+} e^{\ln\left[(1+x)^{\frac{1}{x}}\right]} = \lim_{x\to 0^+} e^{\frac{1}{x}\cdot\ln(1+x)} =$$

$$= e^{\lim_{x\to 0^+} \frac{1}{x}\ln(1+x)}.$$

Now

$$\lim_{x\to 0^+} \frac{1}{x}.\ln(1+x) = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} \underset{\frac{0}{0} \Rightarrow H.R.}{=} \lim_{x\to 0^+} \frac{\frac{1}{1+x}}{1} = \frac{1}{1} = 1$$

$$\Rightarrow \lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e^{\lim_{x\to 0^+} \frac{1}{x}\ln(1+x)} = e^1 = e$$

Note that the result of this limit is not surprising since the number e is defined as:

$$e = \lim_{x\to 0} (1+x)^{\frac{1}{x}}.$$