ANTIDERIVATIVES (Sec. 4.7)

The process of differentiation consists in computing the derivative of a given function. Can this process be reversed, that is, is it possible to find a function F whose derivative is a known function f?

If such a function F exists, it is called an "antiderivative" of f.

Def: A function F is called antiderivative of f on an interval I if $F'(x) = f(x)$, for all x in I.

ex: An antiderivative of the function $f(x) = 2x$ is $F(x) = x^2$. Indeed $F'(x) = 2x$.

Note that the antiderivative of a function is not unique (and this is the reason why we use the indefinite article "an")

In our example also the functions x^2+1, $x^2+\sqrt{2}$, x^2+2018 etc. are antiderivatives of $f(x) = x^2$, because the derivative of a constant is zero.

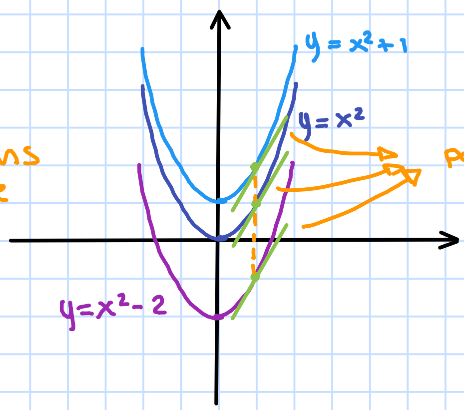
$$x^2+1 \qquad x^2+\sqrt{2} \qquad x^2+2018$$

differentiation

$$2x$$

Geometrically this corresponds to the fact that if we shift the graph of a function vertically, this does not affect the slopes of the tangent lines to the graph at each point.

All these functions have the same derivative



parallel lines = same slope

This means that if $F(x)$ is an antiderivative of a function $f(x)$ then, for all c in $\mathbb{R}$, also $F(x)+c$ is an antiderivative of $f(x)$. Indeed:

$$[F(x)+c]' = F'(x) + 0 = F'(x) = f(x)$$

Now assume that two functions $F(x)$ and $G(x)$ have the same derivative, that is

$$F'(x) = G'(x) \text{ for all } x.$$

Then, if we consider the function $H(x) = F(x) - G(x)$ we have

$$H'(x) = F'(x) - G'(x) \underset{\uparrow}{=} 0 \quad \text{for all } x$$

$F'(x) = G'(x)$ for all x

This implies that $H(x)$ is a constant function, that is there exists c in $\mathbb{R}$ such that

$$H(x) = F(x) - G(x) = c$$
$$\Downarrow$$
$$F(x) = G(x) + c$$

We have then the following theorem:

Theorem: If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + c$$

where c is an arbitrary constant.

Because of the previous theorem, once we know a particular antiderivative $F(x)$, then the most general antiderivative is given by $F(x)+c$.

ex: The most general antiderivative of $f(x) = 2x$ is $x^2 + c$, with c in $\mathbb{R}$.

TABLE

FUNCTION	MOST GENERAL ANTIDERIVATIVE
x^n	$\frac{x^{n+1}}{n+1} + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
e^x	$e^x + c$
$\cos(x)$	$\sin(x) + c$

note that this function is not defined when $n=-1$, reason why we consider $\frac{1}{x}$ separately...

FUNCTION	MOST GENERAL ANTIDERIVATIVE
$\sin(x)$	$-\cos(x) + c$
$\sec^2(x)$	$\tan(x) + c$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x) + c$
$\frac{1}{1+x^2}$	$\tan^{-1}(x) + c$

In the previous table note that if we derive the most general antiderivative we get the function on its left-hand.

A quick remark on the most general antiderivative of $\frac{1}{x}$.

The function $f(x) = \frac{1}{x}$ is defined over $(-\infty, 0) \cup (0, \infty)$, so $\ln(x)$ is not an antiderivative of $f(x)$, because it is only defined on $(0, \infty)$.

Let us consider the function $F(x) = \ln|x| = \begin{cases} \ln(x), & x>0 \\ \ln(-x), & x<0 \end{cases}$.

- for $x>0$ we have $(\ln|x|)' \overset{x>0}{=} (\ln(x))' = \frac{1}{x}$
- for $x<0$ we have $(\ln|x|)' \overset{x<0}{=} (\ln(-x))' = \frac{1}{-x} \cdot (-1) = \frac{1}{x}$

Therefore, for all x $(\ln|x|)' = \frac{1}{x}$ and $\ln|x|$ is a particular antiderivative of $\frac{1}{x}$.

Remark: If $F(x)$ is an antiderivative of $f(x)$ and $G(x)$ is an antiderivative of $g(x)$ then:

- if $c \in \mathbb{R}$, $cF(x)$ is an antiderivative of $cf(x)$.
- $F(x) + G(x)$ is an antiderivative of $f(x)+g(x)$.

Exercise: (1) Find the most general antiderivative of

$$g(x) = 3\cos(x) + 2e^x + \frac{-1 + 4x^6 + 2\sqrt[3]{x}}{x}.$$

(2) Find the function $G(x)$ such that $G'(x) = g(x)$ and $G(1) = 2$.

Solution

(1) First we rewrite $g(x)$:

$$g(x) = 3\cos(x) + 2e^x - \frac{1}{x} + \frac{4x^6}{x} + \frac{2\sqrt[3]{x}}{x} =$$

$$= 3\cos(x) + 2e^x - \frac{1}{x} + 4x^5 + \frac{2x^{\frac{1}{3}}}{x} =$$

$$= 3\cos(x) + 2e^x - \frac{1}{x} + 4x^5 + x^{-\frac{2}{3}}$$

So the most general antiderivative of $g(x)$ is:

$$G(x) = 3\sin(x) + 2e^x - \ln|x| + 4\frac{x^6}{6} + 2\frac{x^{-\frac{2}{3}+1}}{-\frac{2}{3}+1} + c =$$

$$= 3\sin(x) + 2e^x - \ln|x| + \frac{2}{3}x^6 + 6x^{\frac{1}{3}} + c$$

(2) The most general antiderivative $G(x)$ computed in (1) is a function such that $G'(x) = g(x)$.

Moreover $G(x)$ has to satisfy

$$G(1) = 2$$

$$\Downarrow$$

$$3\sin(1) + 2e^1 - \ln|1| + \frac{2}{3}\cdot 1^6 + 6\cdot 1^{\frac{1}{3}} + c = 2$$

$$\Downarrow$$

$$3\sin(1) + 2e + \frac{2}{3} + 6 + c = 2$$

$$\Downarrow$$

$$3\sin(1) + 2e + \frac{20}{3} + c = 2$$

solve for c

$$\Downarrow$$

$$c = 2 - 3\sin(1) - 2e - \frac{20}{3} = -\frac{14}{3} - 3\sin(1) - 2e$$

In conclusion:

$$G(x) = 3\sin(x) + 2e^x - \ln|x| + \frac{2}{3}x^6 + 6\sqrt[3]{x} \underbrace{- \frac{14}{3} - 3\sin(1) - 2e}_{c}.$$

The previous exercise shows that, given an additional condition (in our case $G(1) = 2$), there exists only one antiderivative that satisfies that condition.

Indeed the condition results in a linear equation on the constant c, which has one unique solution!

Exercise: A particle moves in a straight line and has acceleration given by

$$a(t) = 4t + 3.$$

Its initial velocity is $v(0) = 3$ cm/s and its initial displacement is $s(0) = 8$ cm. Find its position function $s(t)$.

Solution

We have $a(t) = v'(t) = s''(t)$.

The problem can then be reformulated in the following way:

Find a function $s(t)$ such that

- $s''(t) = 4t+3$
- $v(0) = s'(0) = 3$
- $s(0) = 8$

- $s''(t) = 4t+3 \xRightarrow{\text{COMPUTE ANTIDERIVATIVE}} v(t) = s'(t) = 4\frac{t^2}{2} + 3t + c = 2t^2+3t+c$

$$\Rightarrow \begin{cases} v(t) = 2t^2+3t+c \\ \underbrace{v(0)}_{= c} = 3 \end{cases} \Rightarrow c = 3 \Rightarrow v(t) = 2t^2+3t+3$$

- $v(t) = s'(t) = 2t^2+3t+3 \xRightarrow{\text{COMPUTE ANTIDERIVATIVE}} s(t) = 2\frac{t^3}{3} + 3\frac{t^2}{2} + 3t + d$

$$\Rightarrow \begin{cases} s(t) = \frac{2}{3}t^3 + \frac{3}{2}t^2 + 3t + d \\ \underbrace{s(0)}_{= d} = 8 \end{cases} \Rightarrow d = 8 \Rightarrow s(t) = \frac{2}{3}t^3 + 3\frac{t^2}{2} + 3t + 8$$

Exercise: A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 288 ft above the ground.

(1) Find its height above the ground t second later.

(2) When does it reaches its maximum height?

(3) When does it hit the ground?

Solution

We set:

$h(t)$: position function of the ball (height of the ball at a time t)

$v(t)$: velocity of the ball

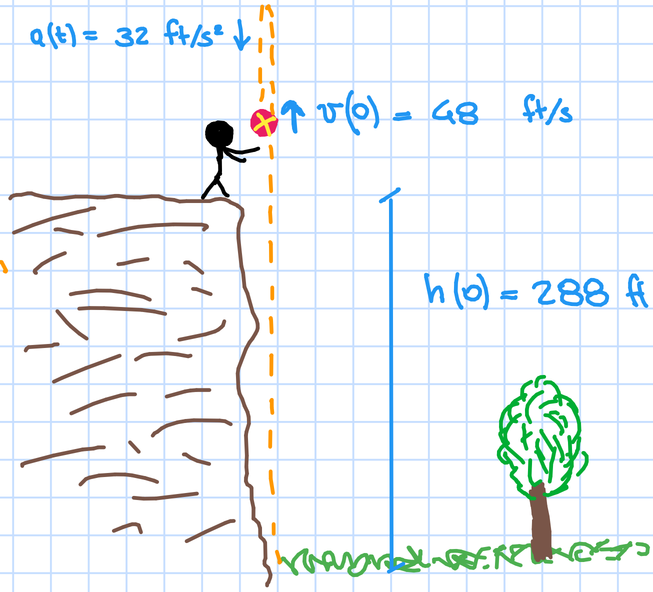
$a(t)$: acceleration of the ball

We have:

$$h''(t) = a(t) = -9.8 \ m/s^2 = -32 \ ft/s^2$$

here the acceleration is the gravity

note that the acceleration is negative because the acceleration is in the opposite direction as the velocity



$$v(0) = h'(0) = 48 \ ft/s$$

$$h(0) = 288 \ ft$$

(1) • $a(t) = -32 \underset{\text{find antiderivative}}{\Rightarrow} \begin{cases} v(t) = -32t + c \\ v(0) = 48 \end{cases} \Rightarrow \begin{cases} v(t) = -32t + c \\ c = 48 \end{cases} \Rightarrow v(t) = -32t + 48$

$$v(0) = c$$

• $v(t) = -32t + 48 \underset{\text{find antiderivative}}{\Rightarrow} \begin{cases} h(t) = -32 \cdot \frac{t^2}{2} + 48t + d \\ h(0) = 288 \end{cases} \Rightarrow \begin{cases} h(t) = -16t^2 + 48t + d \\ d = 288 \end{cases} \Rightarrow$

$$h(0) = d$$

$$\Rightarrow h(t) = -16t^2 + 48t + 288$$

The function $h(t)$ represents the height of the ball above the ground t seconds later.

(2) We are asked to find the absolute maximum value of $h(t)$. Of course this happens when the velocity of the ball is zero (indeed at that moment the ball reaches the peak and starts going down).

$$v(t) = h'(t) = -32t + 48 = 0 \Rightarrow t = \frac{48}{32} = \frac{3}{2} \ s$$

$$\Rightarrow h\left(\frac{3}{2}\right) = -16 \cdot \left(\frac{3}{2}\right)^2 + 48 \cdot \frac{3}{2} + 288 = 324 \ ft$$

(3) We want to solve the equation $h(t) = 0$.

$$-16t^2 + 48t + 288 = 0$$
$$\Downarrow$$
$$-16(t^2 - 3t - 18) = 0$$
$$\Downarrow$$
$$-16(t-6)(t+3) = 0 \Rightarrow t = 6s \text{ or } \cancel{t = -3} \ s.$$

time can not be negative